Math 314 Section 2: Homework 4 Solutions

1. Problem 1:

   (a) Do problem 7.3.7.

   (b) Suppose there are m boys and n girls at the party. How many edges would the graph representing their dancing have? What are its degrees?

   Solution: Omitted. For (a), draw $K_{6,7}$. For (b), its $K_{m,n}$, which has mn edges.

2. Problem 7.3.12. Let G be a connected graph in which every pair of edges have an endpoint in common. Show that G is either a star or K_3.

   Solution: If G has zero edges, its just K_1, which is a star. If G has one edge, its K_2, which is a star. If G has more than 2 edges, let e and f be two of them. Suppose e and f share a vertex v. If e and f are the only two edges, we have a star. If not, then there is a third edge, g.

   If g connects the other endpoints of e and f, we have a K_3. In this case, we cannot add a 4th edge, since any additional fourth edge must touch exactly one of e, f, or g, producing two edges that do not share a vertex.

   If g does not connect the other endpoints of e and f, then g must touch v. In this case, we can add as many additional edges as we like, as long as they all touch v and nothing else. This creates a star.

3. Problem 7.3.10. Let G be connected with at least two vertices. Prove that there is a vertex of G whose removal does not disconnect G.

   Solution: Since G is connected, it contains a spanning tree T. Since T is a tree, it contains a degree 1 vertex, v. If we delete v from T, the remaining graph $T\backslash v$ is still a tree (deleting a leaf from a tree gives another tree). In fact, $T\backslash v$ is a spanning tree for $G\backslash v$, and hence $G\backslash v$ is connected (to find a path between any two vertices of $G\backslash v$, simply take the unique path between them in $T\backslash v$).

4. Problem 8.5.2. Let G be connected. Prove that e is not a cut-edge if and only if e is contained in a circle in G.

Solution: There are two statements to prove, the "if" and the "only if".

First, suppose that e is not a cut-edge, and let u and v be the endpoints of e. Since e is not a cut edge, $G \backslash e$ is connected. Thus, there is a path P connecting u and v in $G \backslash e$. So, $P \cup e$ is a cycle in G that contains e.

Conversely, suppose that e (with endpoints u and v) is contained in a cycle in G. We want to show that removing e does not disconnect the graph. Suppose that two nodes x and y of G are connected by a path P. If P does not contain e, then x and y are still connected when we delete e. If P does contain e, then P is destroyed when we delete e. However, we can still find a path from x to y—first, take a path from x to u, then a path from u to v (the other way around C), then a path from v to y. This implies that there is a path from x to y (by Page 131). So, e is not a cut-edge.

5. In a graph G, an *independent set* is a subset I of the vertices of G such that there is no edge of G with both endpoints in I. The empty set counts as an independent set.

   Recall that P_n is the path graph with n vertices. Find a recursion for the number of independent sets in P_n. Can you find a recursion for C_n (the cycle with n vertices) as well?

   Solution: First check that P_1 has 2 independent sets and P_2 has 3 independent sets. In general, let I_n be the number of independent sets contained in P_n. We will get a recursive formula for I_n.

   Let I be an independent set in P_n. Let v be a degree 1 vertex of P_n, and let w be its single neighbor.

   If I contains v, then it cannot contain w. Then, the rest of I (ignoring v and w) consists of any independent set taken from $P_n \backslash \{v, w\}$, which is a P_{n-2}. Thus in this case the number of choices for I is I_{n-2}.

   If I does not contain v, then I consists of any independent set taken from $P_n \backslash v$, which is a P_{n-1}. Thus, the number of choices for I is I_{n-1}.

   Therefore, the total number of choices for I is $I_n = I_{n-1} + I_{n-2}$. Combined with the fact that $I_1 = 2$ and $I_2 = 3$, we see that $I_n = F_{n+2}$ (F_n are the Fibonacci numbers).

For a cycle, it is very similar. Let J_n be the number of independent sets in C_n. Let v be a particular vertex of C_n with neighbors x and y. If an independent set I contains v, it cannot contain x or y. The rest of the independent set is taken from $C_n \setminus \{x, v, y\}$ which is a P_{n-3}. If I does not contain v, then I consists of an independent set taken from $C_n \setminus \{v\}$, which is a P_{n-1}. So the total count is $J_n = I_{n-3} + I_{n-1} = F_{n-1} + F_{n+1}$. These are the Lucas numbers.