Math 314 Section 2: Homework 2 solutions

1. **[2.5.2]** What is the following sum?

$$0 \cdot \binom{n}{0} + 1 \cdot \binom{n}{1} + ...(n-1) \cdot \binom{n}{n-1} + n \cdot \binom{n}{n}$$

   Solution: The sum is $n2^{n-1}$.

   Proof: This can be proved by counting, induction, the binomial theorem, or just straight algebra. Here is a proof by algebra.

   First, notice that

$$k \cdot \binom{n}{k} = k \cdot \frac{n!}{k!(n-k)!} = \frac{n!}{(k-1)!(n-k)!} = n\frac{(n-1)!}{(k-1)!(n-k)!} = n\binom{n-1}{k-1}.$$

   Apply the above identity to every term in the main expression:

$$\begin{aligned} 0 \cdot \binom{n}{0} + 1 \cdot \binom{n}{1} + ...(n-1) \cdot \binom{n}{n-1} + n \cdot \binom{n}{n} &= \\ 0 + n \cdot \binom{n-1}{0} + n \cdot \binom{n-1}{1} + ... + n \cdot \binom{n-1}{n-1} &= \\ n \cdot \left[\binom{n-1}{0} + \binom{n-1}{1} + ... + \binom{n-1}{n-1}\right] &= \\ n2^{n-1} & \end{aligned}$$

2. **[2.5.4(b)]** Prove that $n^3 - n$ is a multiple of 6 for all n.

   Proof 1: Induction on n. For the base case, let $n = 0$. Then, $0^3 - 0 = 0$, which is a multiple of 6.

   Assume the statement is true for a given n. In other words, assume that $n^3 - n$ is divisible by 6.

   Then, $(n+1)^3 - (n+1) = n^3 + 3n^2 + 3n + 1 - (n+1) = (n^3 - n) + 3(n^2 + n)$. By the induction assumption, $(n^3 - n)$ is divisible by 6. Furthermore, $3(n^2 + n) = 3(n)(n+1)$ is also divisible by 6, since either n or $n+1$ is even (they are consecutive integers). Therefore, $(n+1)^3 - (n+1)$ is divisible by 6.

   Proof 2: By algebra, $n^3 - n = (n-1)(n)(n+1)$. These are three consecutive integers. Therefore, one of them is divisible by 2, and one of them is divisible by 3. Thus, $n^3 - n$ is divisible by 6.

3. **[2.5.5]** There are 40 girls. There are 18 who like chess, 23 who like soccer, and an unknown number who like biking. The number who like chess and soccer is 9. The number who like chess and biking is 7. The number who like soccer and biking is 12. There are 4 girls who like all 3 activities. How many like biking?

   Solution: Inclusion/Exclusion. Let C be the set of those who like chess, S for soccer, and B for biking. By inclusion/exclusion:

$$|C \cup B \cup S| = |C| + |S| + |B| - |C \cap S| - |C \cap B| - |S \cap B| + |C \cap S \cap B|.$$

   Substituting the numbers, we get $40 = 18 + 23 + x - 9 - 7 - 12 + 4$, which means $x = 23$.

4. **[3.4.1/3.8.10]** How many ways are there to distribute n pennies among k children so that each child gets at least 2 (or, each child gets at least 5)?

   Solution: We will solve the problem involving 5 pennies. The solution for 2 pennies is almost identical.

   First, since each child must get at least 5 pennies, we simply give each child 5 pennies. There is one way to do this. After doing this, we have $n - 5k$ pennies remaining. We wish to distribute these $n - 5k$ pennies among the k children, and we now have no restrictions as to how many to give each child. So, the number of ways is

$$\binom{(n-5k)+k-1}{k-1} = \binom{n-4k-1}{k-1}.$$

5. **[3.8.12]**Prove the following:

$$1 + \binom{n}{1}2 + \binom{n}{2}4 + ... + \binom{n}{n-1}2^{n-1} + \binom{n}{n}2^n = 3^n.$$

   Proof 1: Counting. We claim that both sides count the number of strings of length n using elements from $S = \{1, 2, 3\}$. This is clearly true for the right hand side.

   For the left hand side, suppose a string A has $n - k$ 1's in it. Since A has length n, there are $\binom{n}{n-k}$ ways to decide where to put the 1's. After placing the 1's, we must choose whether every other entry in the sequence is a 2 or a 3. Since there are k entries remaining and 2 choices for each, there are 2^k choices for how to place the 2's and 3's. So, the number of possibilities for A is $\binom{n}{n-k}2^k = \binom{n}{k}2^k$. Do this for every possible k and add up the terms to get the left hand side.

   Proof 2: Apply the binomial theorem with $x = 1$ and $y = 2$. [Try this!!]

   Proof 3: Both sides count the number of ways to select a k-subset K from an n-set N, and then select any subset of K. [Also try this :)]

6. **[Extra]** Suppose that you choose 100 different numbers from the set $S = \{1, 2, ..., 197\}$. Prove that you can always find two of your chosen numbers that sum to 198.

   Proof: Break the numbers in S into groups as follows:

   $$(1, 197), (2, 196), (3, 195), ..., (97, 101), (98, 100), (99).$$

   Ignoring the group (99), there are a total of 98 groups. By design, the two numbers in each group sum to 198. Since we are selecting 100 numbers from S and we have 98 groups, by the pigeonhole principle two of our numbers must be in the same group and hence sum to 198.