Math 314 Section 2: Homework 6 Solutons

1. Problem 12.3.4

   Since the graph is bipartite, each face has even length (at least 4). Thus, $4f \leq 2e$, which means $f \leq (e/2)$. Plugging this into Euler's formula, we get

   $$2 = v - e + f \leq v - e + (e/2).$$

   Rearrange to obtain $e \leq 2v - 4$.

2. Solve this modified version of problem 12.3.6:

   (a) First, let P be a convex polyhedron (3-dimensional) with v vertices, e edges, and f faces. Assume that every vertex of P has degree 3. Deduce that $3v = 2e$.

   Every vertex is degree 3, and every edge touches two vertices. Hence $3v = 2e$.

   (b) Use part (a) to deduce that $f = 2 + \frac{1}{3}e$.

   Plug $v = (2/3)e$ (from part (a)) into Euler's formula and solve for f.

   (c) Suppose that P has *only* pentagonal and hexagonal faces. Let p be the number of pentagonal faces and h the number of hexagonal faces. Prove that $p = 12$.

   Since all faces are pentagonal or hexagonal, $p + h = f$. Each pentagonal face touches 5 edges and each hexagonal face touches 6. This counts all the edges twice. So, $5p + 6h = 2e$. Substituting this into the formula from (b), we get:

   $$p + h = 2 + \frac{1}{3}\frac{5p + 6h}{2}.$$

   Solve for p to get $p = 12$.

   (d) For fun, give an example of a specific P that fits the description of part (c).

   A soccer ball!

3. Problem 13.4.5. Hint: Its convenient to think of the graph as the complement of C_n. Hint 2: The answer is $\lceil \frac{n}{2} \rceil$. So, there are two things to show: first, you can always color it with $\lceil \frac{n}{2} \rceil$ colors, second, you can't color it with fewer colors.

   Solution: Let the vertices of the removed Hamilton cycle be in order $v_1, v_2, ..., v_n$. Then, v_1 and v_2 are non-adjacent and can be colored the same. Similarly for v_3 and v_4, etc. This gives a coloring with $\lceil \frac{n}{2} \rceil$ colors (one vertex will have its own color if n is odd).

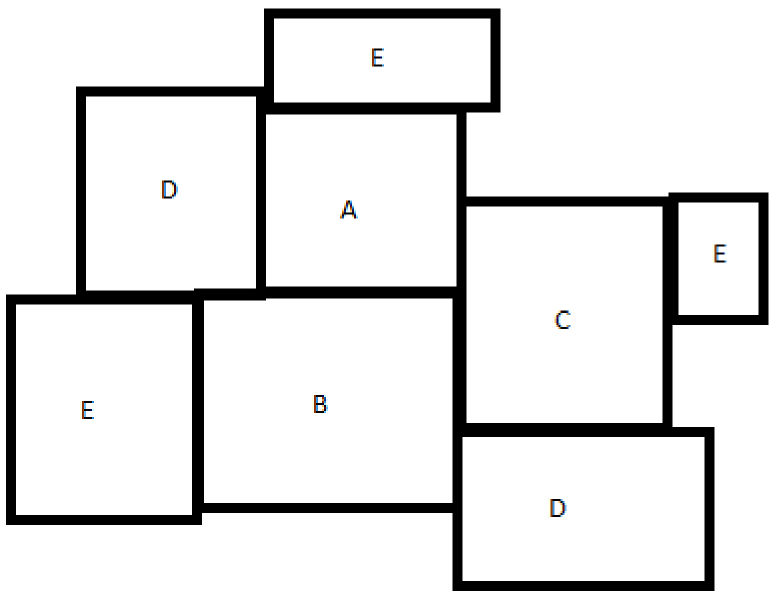
   Now we must check that we can't color G_n with less than $\lceil \frac{n}{2} \rceil$ colors. To see this, notice that there will be at least one edge among any three vertices chosen. Thus, it is impossible to have three vertices of the same color. So the best we could possibly do is to have as many groups of two vertices colored the same as possible, which occurs in the coloring described above.

4. Let G be the graph obtained from two 3-cycles by adding a single edge bewteen them (so G has 6 vertices and 7 edges). Use the deletion-contraction recurrence to calculate the chromatic polynomial for G.

   See link on class page.

5. Show that the 4-color theorem no longer holds if you allow disconnected countries.

   Here is a rather ugly example (countries are A, B, C, D, E):

   Every country touches every other country, so at least 5 colors are needed.