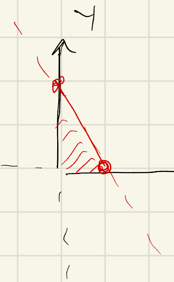
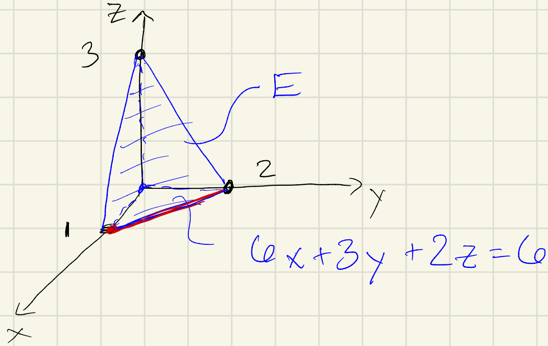
1. $\iiint_E x^2 \, dV$





$0 \le y \le -2x+2$

$0 \le x \le 1$

$\underbrace{6x+3y=6}$

$\Downarrow$

$y = -2x+2$

$0 \le z \le (6-6x-3y)/2$

$$\therefore \boxed{\int_0^1 \int_0^{-2x+2} \int_0^{(6-6x-3y)/2} x^2 \, dz \, dy \, dx}$$

2. $x = \rho \sin\varphi \cos\theta$
$y = \rho \sin\varphi \sin\theta$
$z = \rho \cos\varphi$

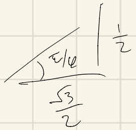
$$\cancel{\rho}\cos\varphi = z = -\sqrt{x^2+y^2} = -\cancel{\rho}|\sin\varphi|$$

$$\therefore \frac{|\sin\varphi|}{\cos\varphi} = -1 \Rightarrow \varphi = 3\pi/4$$

$$\cancel{\rho}\cos\varphi = z = \sqrt{3x^2+3y^2} = \sqrt{3}\,(\cancel{\rho}|\sin\varphi|)$$

$$\therefore \frac{|\sin\varphi|}{\cos\varphi} = \frac{1}{\sqrt{3}} \Rightarrow \varphi = \pi/6$$

Hence, $\int_{\pi/6}^{3\pi/4} \int_0^{2\pi} \int_0^2 \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$



$$= 2\pi \left(\int_{\pi/6}^{3\pi/4} \sin\varphi \, d\varphi\right)\left(\int_0^2 \rho^2 \, d\rho\right)$$

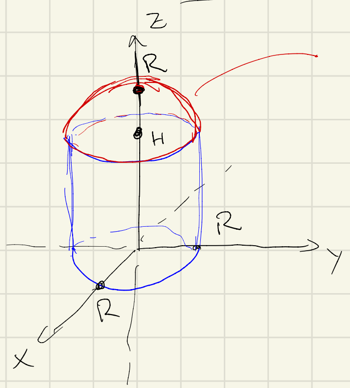
$$= 2\pi\left(-\cos\varphi \Big|_{\pi/6}^{3\pi/4}\right)\left(\frac{1}{3} \rho^3\Big|_0^2\right)$$

$$= 2\pi\left(-\left[-\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\right]\right)\left(\frac{8}{3}\right)$$

$$= 2\pi\left(\frac{\sqrt{2}+\sqrt{3}}{2}\right)\left(\frac{8}{3}\right)$$

$$= \boxed{\frac{\pi(\sqrt{2}+\sqrt{3})\,8}{3}}$$

3. (a)



$$x^2+y^2+(z-H)^2=R^2$$

$$r^2+(z-H)^2=R^2$$

$$f(r,\theta)=z=\sqrt{R^2-r^2}+H$$

(b) $$\int_0^{2\pi}\int_0^R\int_0^H \underset{(1)}{r}\, dz\, dr\, d\theta + \int_0^{2\pi}\int_0^R\int_H^{\sqrt{R^2-r^2}+H} \underset{(2)}{r}\, dz\, dr\, d\theta$$

(c) (1) $$\left(\int_0^{2\pi} d\theta\right)\left(\int_0^H dz\right)\left(\int_0^R r\, dr\right)$$

$$= (2\pi)(H)\left(\frac{1}{2}R^2\right) = H\pi R^2 \quad \checkmark$$

(2) $$\left(\int_0^{2\pi} d\theta\right)\left(\int_0^R r\left[\sqrt{R^2-r^2}\right] dr\right)$$

$$= (2\pi)\left(-\frac{1}{2}\int_{R^2}^0 u^{\frac{1}{2}}\, du\right)$$

$$= +\pi\int_0^{R^2} u^{\frac{1}{2}}\, du$$

$$= \pi\left[\frac{2}{3}u^{3/2}\Big|_0^{R^2}\right]$$

$$= \frac{2}{3}\pi R^3 \quad \checkmark$$

$u = R^2 - r^2$

$du = -2r\, dr$

$-\frac{1}{2}du = r\, dr$

4. $\vec{r}(x) = \langle x, x^2, x^3 \rangle \qquad x \in [0,1]$

$h(x) = 1 + 4y + 9xz$

(a) $\int_C h \, ds = \int_0^1 (1+4x^2+9x^4)\sqrt{1+4x^2+9x^4} \, dx$

(b) $\int_C h \, dy = \int_0^1 (1+4x^2+9x^4) \, 2x \, dx$

$= \int_0^1 2x + 8x^3 + 18x^5 \, dx$

$= x^2 + 2x^4 + 3x^6 \Big|_0^1$

$= 1 + 2 + 3$

$= 6$

5. $\vec{r}(t) = \langle a \sin t, b \cos t, ct \rangle, \qquad t \in [0, \pi/2]$

$\vec{r}\,'(t) = \langle a \cos t, -b \sin t, c \rangle$

$\vec{r}\,''(t) = \langle -a \sin t, -b \cos t, 0 \rangle$

$\vec{F} = \langle -am \sin t, -bm \cos t, 0 \rangle$

$W = \int_C \vec{F} \cdot d\vec{r}$

$= \int_0^{\pi/2} \langle -am \sin t, -bm \cos t, 0 \rangle \cdot \langle a \cos t, -b \sin t, c \rangle \, dt$

$= \int_0^{\pi/2} -a^2 m \sin t \cos t + b^2 m \cos t \sin t \, dt$

$= (b^2 - a^2) m \int_0^{\pi/2} \sin t \cos t \, dt \qquad u = \sin t$

$\qquad du = \cos t \, dt$

$= (b^2 - a^2) m \left[\int_0^1 u \, dt \right]$

$= \dfrac{(b^2 - a^2) m}{2}$

6. $F = \langle \underbrace{z\sin y}_{P}, \underbrace{xz\cos y}_{Q}, \underbrace{x\sin y}_{R} \rangle$; domain $\vec{F} = \mathbb{R}^3$

$$\text{curl } F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z\sin y & xz\cos y & x\sin y \end{vmatrix}$$

$$= \langle x\cos y - x\cos y, -(\sin y - \sin y), z\cos y - z\cos y \rangle$$

$$= \vec{0}$$

Since curl $\vec{F} = \vec{0}$ and $\vec{F}$ is defined on $\mathbb{R}^3$, $\vec{F}$ is conservative. Hence, $\int_C \vec{F} \cdot d\vec{r}$ is independent of paths.

find f s.t. $\nabla f = \vec{F}$:

$f_x = z\sin y \Rightarrow f(x,y,z) = xz\sin y + g(y,z)$

$$\Downarrow$$

$\boxed{f_y = xz\cos y}$ $\qquad f_y = xz\cos y + g_y(y,z)$

$\boxed{f_z = x\sin y}$ $\qquad = xz\cos y$

$\therefore g_y(y,z) = 0 \Rightarrow g(y,z) = h(z)$

$$\therefore\ f(x,y,z) = xz\sin y + h(z)$$

$$\Rightarrow f_z = x\sin y + h'(z)$$

$$= x\sin y$$

$$\Rightarrow h(z) = K \sim \text{constant}$$

$$f(x,y,z) = xz\sin y + K$$

Hence, $\int_C \vec{F}\cdot d\vec{r} = f(3, \pi/2, 4) - f(0, \pi, 1)$

$$= 3\cdot 4 \sin\left(\tfrac{\pi}{2}\right) - 0$$

$$= \boxed{12}$$

7.

$$\oint_C y^2\,dx + 2e^x\,dy = \iint_D 2e^x - 2y \; dA$$

$$= \int_2^3 \int_1^4 2e^x - 2y \; dy\,dx$$

$$= \int_2^3 2ye^x - y^2 \Big|_1^4 \; dx$$

$$= \int_2^3 (8e^x - 16) - (2e^x - 1) \; dx$$

$$= \int_2^3 6e^x - 15 \; dx$$

$$= 6e^x - 15x \Big|_2^3$$

$$= (6e^3 - 45) - (6e^2 - 30)$$

$$= 6e^3 - 6e^2 - 15$$

8. (a) To show $\vec{F}$ is not conservative,

it suffices to show that

$\text{curl}\,\vec{F} \neq \vec{0}$.

So, $\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & zy^2 & yz \end{vmatrix}$

$= \langle \underbrace{z - y^2}, \ldots \rangle$

we can stop here since

$z-y^2$ is not 0 for every

value of y, z in the domain.

(b) Suppose it is: $\vec{G} = \text{curl}\,\vec{F}$ for some

vector field $\vec{F}$.

Then, $2x + 2x + y = \text{div}\,\vec{G} = \text{div curl}\,\vec{F} = 0$, a

contradiction.