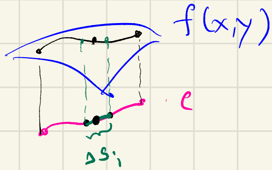
16.7: Surface Integrals

Recall, to calculate $\int_C f(x,y)\,ds$, it was useful to parametriz C, say with $\vec{r}(t) = \langle x(t), y(t) \rangle$, $t \in [a,b]$:

$$\int_C f(x,y)\,ds = \int_a^b f(\vec{r}(t))\,\underbrace{|\vec{r}\,'(t)|}_{}\,dt$$

came from the arc lengths of arbitrarily small subarcs of C

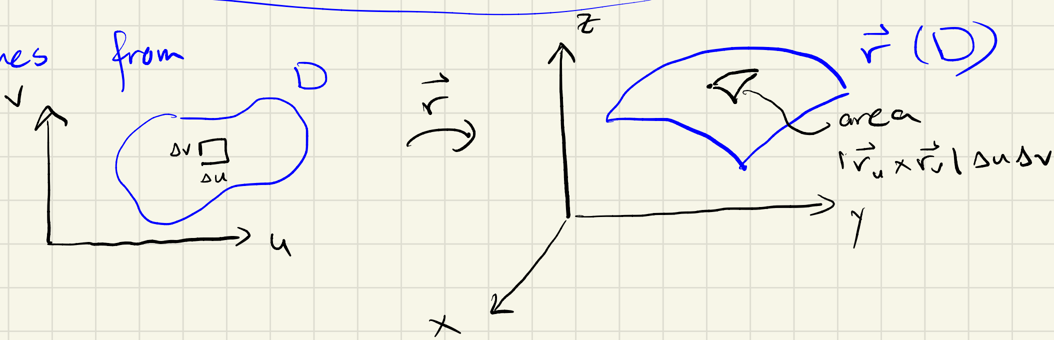


Let S be a surface given by $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$ where $(u,v) \in D \subseteq \mathbb{R}^2$.

The surface integral of f of x,y,z over S is

$$\iint_S f(x,y,z)\,dS := \iint_D f(\vec{r}(u,v))\,\underbrace{|\vec{r}_u \times \vec{r}_v|}_{}\,dA.$$

comes from



Remark: If $f(x,y,z) = 1$, then

$$\iint_S 1\,dS = \iint_D |\vec{r}_u \times \vec{r}_v|\,dA = \text{surface area of } S$$

If S is the graph of $z=g(x,y)$ where $(x,y)\in D$, then we saw last time that we can paramatrize S with $\vec{r}(x,y) = \langle x, y, g(x,y)\rangle$ and $|\vec{r}_x \times \vec{r}_y| = \sqrt{(g_x)^2+(g_y)^2+1}$

$$\therefore \iint_S f(x,y,z)\, dS = \iint_D f(x,y,g(x,y))\sqrt{(g_x)^2+(g_y)^2+1}\, dA.$$

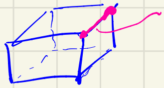
This definition can be altered if $y=g(x,z)$ or $x=g(y,z)$.

To integrate over a vector field, we first need the following:

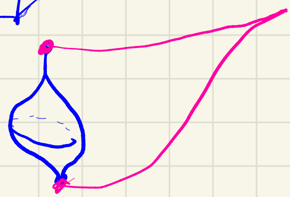
Def: Let S be a surface s.t. there is a tangent plane at every pt. of S (there is a normal vector at every pt. of S).

If there is a cts. choice of normal vectors for each pt. (i.e. we can define a cts. vector field on S), then we say S is orientable.

- the surface has no creases, e.g.



there is no tangent plane



so we want "smooth-looking" surfaces

e.g. 

· not orientable – Möbius strip.

If S is a smooth, orientable surface given by $\vec{r}(u,v)$, then normal vectors have the following form:

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

vector field and it's cts. because it is orientable

ex) What is the form of a normal vector of $x^2+y^2+z^2=a^2$?

<u>Sol:</u> $\vec{r}(\theta, \varphi) = \langle a\cos\theta\sin\varphi,\ a\sin\theta\sin\varphi,\ a\cos\varphi \rangle$

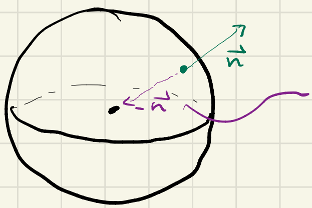
where $0 \le \theta \le 2\pi$ and $0 \le \varphi \le \pi$.

It can be calculated that

$$\vec r_\varphi \times \vec r_\theta = \langle a\sin^2\varphi\cos\theta,\ a^2\sin^2\varphi\sin\theta,\ a^2\sin\varphi\cos\varphi\rangle$$

and $|\vec r_\varphi \times \vec r_\theta| = a^2\sin\varphi$.

$\therefore\ \vec n(\theta,\varphi) = \langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi\rangle.$



$$\frac{\vec r_\theta \times \vec r_\varphi}{|\quad \# \quad|}$$

__Convention:__ For a closed region (a surface that is the boundary of some region E), positive orientation of the surface is the choice of normal vectors which point outward from E. Inward = negative orient.

__Def:__ If $\vec F$ is a cts. vector field defined on an oriented surface S, then the surface integral of $\vec F$ over S

$$\iint_S \vec F \cdot d\vec S := \iint_S \vec F \cdot \vec n \, dS$$

given by orientation

called the flux of $\vec{F}$ across S.

If S is given by $\vec{r}(u,v)$, then

$$\iint_S \vec{F}\cdot d\vec{S} := \iint_S \vec{F}\cdot\left(\frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}\right)\, dS$$

$$= \iint_D \vec{F}(\vec{r}(u,v))\cdot\left(\frac{\vec{r}_u\times\vec{r}_v}{\cancel{|\vec{r}_u\times\vec{r}_v|}}\right)\cancel{|\vec{r}_u\times\vec{r}_v|}\, dS$$

$$= \iint_D \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u\times\vec{r}_v)$$

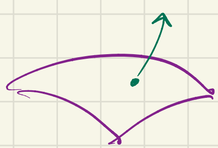
In the case when S is given $z = g(x,y)$ and $\vec{F} = \langle P, Q, R\rangle$ we have

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot(\vec{r}_x\times\vec{r}_y)\, dA$$

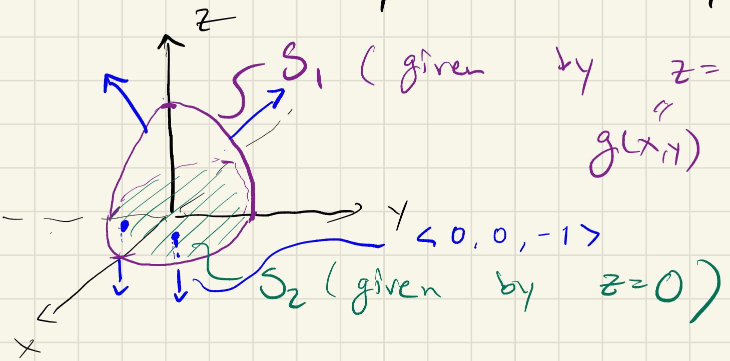
$$= \iint_D \langle P, Q, R\rangle\cdot\left\langle -\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\right\rangle dA$$

$$= \iint_D -P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\, dA$$

formula assume the positive orientation



ex) Evaluate $\iint_S \vec{F} \cdot d\vec{S}$ where $\vec{F} = \langle x, y, z \rangle$ and S is the boundary of the solid region E enclosed by $z = 1 - x^2 - y^2$ and $z = 0$



$$\iint_{S_1} \vec{F} \cdot d\vec{S} = \iint_{x^2+y^2 \leq 1} \langle x, y, 1 - x^2 - y^2 \rangle \cdot \langle 2x, 2y, 1 \rangle \, dA$$

$$= \iint_{x^2+y^2 \leq 1} 2x^2 + 2y^2 + 1 - x^2 - y^2 \, dA$$

$$= \iint_{x^2+y^2 \leq 1} x^2 + y^2 + 1 \, dA$$

$$= \int_0^{2\pi} \int_0^1 (r^2 + 1) r \, dr \, d\theta$$

$$= 2\pi \left(\tfrac{1}{3} r^3 + \tfrac{1}{2} r^2 \Big|_0^1 \right)$$

$$= 2\pi \left(\tfrac{1}{3} + \tfrac{1}{2} \right) = 2\pi \left(\tfrac{5}{6} \right) = \frac{5\pi}{3}$$

$$\iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_{S_2} \vec{F} \cdot \vec{n} \, dS = \iint_{S_2} -z \, dS$$

$$= \iint_{x^2+y^2 \leq 1} 0 \, dS = 0$$

Final answer: $\boxed{5\pi/3}$