16.5 : Curl and Divergence

Def: Consider $\vec{F} = \langle P, Q, R \rangle$ s.t. 1^{st} order partial derivatives of P, Q, R exist. Then the curl of $\vec{F}$ is

$$\text{curl}(\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$$

$$= \langle R_y - Q_z, -(R_x - P_z), Q_x - P_y \rangle.$$

If we denote $\nabla := \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$,

$$\text{curl}(\vec{F}) = \nabla \times \vec{F}.$$

Exercise: If $\vec{F}(x,y,z) = \langle xz, xyz, -y^2 \rangle$, find $\text{curl}\,\vec{F}$.

Thm. 1: If f has cts. 2^{nd}-order partial derivatives on a disk D, then $\text{curl}(\nabla f) = \vec{0}$.

Pf: $\text{curl}\,\nabla f = \nabla \times \nabla f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_x & f_y & f_z \end{vmatrix}$

Reminder: $f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$

$$= \langle f_{zy} - f_{yz}, -(f_{zx} - f_{xz}), f_{yx} - f_{xy} \rangle$$

Clairaut's

$$= \langle 0, 0, 0 \rangle = \vec{0}. \quad //$$

Remark: *If $\vec{F}$ is conservative, then curl $\vec{F} = \vec{0}$.

$\exists$ a fnc. f s.t. $\nabla f = \vec{F}$. So by Thm. 1,

$$\text{curl}\,\vec{F} = \text{curl}\,\nabla f = \vec{0}.$$

The converse is not always true, but...

If curl $\vec{F} \neq \vec{0}$, then $\vec{F}$ is NOT conservative.

Thm. 2: If $\vec{F}$ is a vector field defined on $\mathbb{R}^3$ whose component fnc's have cts. partial derivatives and curl $\vec{F} = \vec{0}$, then $\vec{F}$ is conservative.

Exercise(s): (a) Show that $\vec{F}(x,y,z) = \langle y^2 z^3, 2xyz^3, 3xy^2z^2 \rangle$ is conservative.

(b) Find a fnc. f s.t. $\nabla f = \vec{F}$.

Remark: *If curl $\vec{F} \neq \vec{0}$, then $\vec{F}$ is NOT conservative.

Def: If $\vec{F} = \langle P, Q, R \rangle$ and the 1st order partial derivatives of P, Q, R exist, then the divergence of $\vec{F}$ is

$$\text{div}(\vec{F}) = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Exercise: If $\vec{F}(x,y,z) = \langle xz, xyz, -y^2 \rangle$, find div $\vec{F}$.

Thm. 3: If $\vec{F} = \langle P, Q, R \rangle$ is a vect. field on $\mathbb{R}^3$ and P, Q, R have cts. 2^{nd}-order partial derivatives, then $\operatorname{div}(\operatorname{curl} \vec{F}) = 0$.

Pf: $\operatorname{div} \operatorname{curl} \vec{F} = \nabla \cdot (\nabla \times \vec{F})$

$$= \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle \cdot \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

Clairaut's

$$= \cancel{R_{yx}} - \cancel{Q_{zx}} + \cancel{P_{zy}} - \cancel{R_{xy}} + \cancel{Q_{xz}} - \cancel{P_{yz}}$$

$$= 0. \quad //$$

ex) Show that $\vec{F}(x,y,z) = \langle xy, xyz, -y^2 \rangle$ can't be written as the curl of another vector field.

Sol: By way of contradiction, suppose that we can find vector field $\vec{G}$ s.t. $\vec{F} = \operatorname{curl} \vec{G}$.

Then, $\underset{y + xz}{\underset{\parallel}{\operatorname{div} \vec{F}}} = \operatorname{div} \operatorname{curl} \vec{G} = 0$, a contradiction.

Hence, there does not exist such a vector field.

Rewriting Green's Theorem:

Let C be a curve and P and Q functions satisfying Green's Theorem. Let D be the region enclosed by C. If $\vec{F} = \langle P, Q \rangle$, then

$$\oint \vec{F} \cdot d\vec{r} = \oint P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \; dA$$

but

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & 0 \end{vmatrix}$$

$$= \begin{vmatrix} \partial/\partial x & \partial/\partial y \\ P & Q \end{vmatrix} \hat{k} - \cancelto{0}{\frac{\partial}{\partial z} \begin{vmatrix} \hat{i} & \hat{j} \\ P & Q \end{vmatrix}}$$

$$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k}$$

$$\therefore \oint_C \vec{F} \cdot d\vec{r} = \iint_D \text{curl} \vec{F} \cdot \hat{k} \; dA.$$