Relations (continued)

Example 1: The relation on the set of positive integers defined as: $(x, y) \in R$ if and only if $x|y$ is antisymmetric but not symmetric

Proof.

To show that the relation is antisymmetric, suppose $(a, b), (b, a) \in R$.That is $a|b$ and $b|a$, so that $a = k_1 \cdot b$ and $b = k_2 \cdot a$
This gives $a = k_1 k_2 \cdot a$, and we have $k_1 k_2 = 1$.As a, b are positive integers, we have $k_1 = k_2 = 1$, which implies that $a = b$. This shows that R is anti-symmetric. R is not symmetric since $2|4$ but $4 \nmid 2$ □

Exercises1:

1. How many relations are there on a set with n elements that are symmetric.
2. Let R be a relation on the set of ordered pairs of positive integers such that $((a, b), (c, d)) \in R$ if and only if $a + d = b + c$. Show that the relation R is reflexive, symmetric and transitive.

Other ways of Representing relations (section 9.3)

You are to read about the following in section 9.3

1. Representing relations using matrices
2. Representing relations using digraphs

Exercises2:

1. How many non-zero entries does the matrix representing the relation R on $A = \{1, 2, 3, \ldots, 100\}$ if R is :
   (i) $\{(a, b) | a > b\}$
   (ii) $\{(a, b) | ab = 1\}$
   (iii) $\{(a, b) : a \neq b\}$
2. Section 9.3 Exercise 24, 27

Section 9.5 Equivalence Relations

Definition: A relation R on a set A is said to be an equivalence relation if it is reflexive,symmetric and transitive. **Examples:**

1. The relation R on the set of integers given by: $(a, b) \in R$ if $a - b$ is an integer is na equivalence relations
2. Let m be an integer with $m > 1$. The relation $R = \{(a, b) | a \equiv b(\mod m)\}$ is an equivalence relation.

Exercise 3

1. Show that the relation R consisting of all pairs (x, y) such that x and y are bit strings that agree in their first and third bits is an equivalence relation

Let R be an equivalence relation on a set A, we say two elements a and b are in the same equivalence class if $(a, b) \in R$. We denote by $[a]_R$ the set of elements that are in same equivalence class as a . **Check this:** The equivalence classes $[a], [b]$ are either the equal or disjoint (that is $[a] = [b]$ or $[a] \cap [b] = \emptyset$).
The above property says that the equivalence classes partitions the set A. That is every element of A belongs to only one equivalence class.
Example: The equivalence relation R on the integers defined by $(a, b) \in R$ if $a \equiv b(\mod 4)$ has the following equivalence classes:

- $[0] = \{\dots, -8, -4, 0, 4, 8, \dots\}$
- $[1] = \{\dots, -7, -3, 1, 5, 9, \dots\}$
- $[2] = \{\dots, -6, -2, 2, 6, 10, \dots\}$
- $[3] = \{\dots, -5, -1, 3, 7, 11, \dots\}$

Exercise 4: What are the equivalence classes of the bit strings $010, 1011$ under the equivalence relation in exercise 3.