Binomial Identities

Notation: $\binom{n}{r}$ is (n combination r)
Recall from last class: $(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$
We showed this by determining the coefficient of $x^r y^{n-r}$ in
$(x+y)^n = (x+y)(x+y)\cdots(x+y)$ (n times)
the coefficient of $x^r y^{n-r}$ is just counting number of ways of selecting r $x's$ out of n $x's$

Identities

Show that $k\binom{n}{k} = n\binom{n-1}{k-1}$

Proof.

Let A be a set of size n

- LHS counts elements of the form:
- (a, S) for $a \in S$, $|S| = k$
- we show that the RHS counts the same thing
- First select an $a \in A$ (n choices)
- Then select a $k-1$ subset T of the remaining $n-1$ set,so a set of the above form is $(a, T \cup \{a\})$ ($\binom{n-1}{k-1}$ choices)

□

Exercises

- Show that $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$
- Show that $\binom{2n}{2} = 2\binom{n}{2} + n^2$ (Hint : think of $2n = n + n$)
- Show that $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$ (Hint : start from RHS , what does it count?)
- Show that $\binom{n}{k} \leq \frac{n^k}{2^{k-1}}$

Vandermonde's Identity

Let m, n, r be non-negative integers with r not exceeding either m or n. Then

- $\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$
- Let A, B be two disjoint sets with $|A| = m, |B| = n$.Then $|A \cup B| = m + n$
- There are $\binom{m+n}{r}$ ways of choosing r elements from $A \cup B$
- Another way of selecting r elements is to pick $r - k$ from A and k from B
- There are $\binom{m}{r-k}\binom{n}{k}$
- For $0 \leq k \leq r$
- In total (i.e adding all these possibilities), there are $\sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$ ways

more exercises

- Give a combinatorial proof that : $\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$
- Give a formula for the coefficient of x^k in the expansion of $(x^2 - \frac{1}{x})^{100}$