Vector space
subspace
linear combination

Chapter 5

Vector Spaces

In this chapter we will review some of the material you have already learned in your basic linear algebra course. The main difference is that instead of considering vector spaces with scalars in the field $\mathbb{R}$ of real numbers, we will consider vector spaces with scalars from any arbitrary field F. We will also talk about Zorn's lemma, and use it to prove that any vector space has a basis. Introductory linear algebra courses only prove that a finitely generated vector space has a basis.

5.1 Basic Definitions

02/11/19

Definition 5.1.1. *Vector space*, *subspace*.

Lemma 5.1.1 *Let F be a field, V a vector space over F, $x \in V$ and $\lambda \in F$. If $\lambda x = 0$ then either $\lambda = 0$ or $x = 0$.*

Definition 5.1.2. Let F be a field, V be a vector space over F, and X a subset of V. A *linear combination* of X over F is an expression of the form

$$\lambda_1 x_1 + \cdots + \lambda_n x_n \tag{5.1}$$

where $x_1, \ldots, x_n \in X$ and $\lambda_1, \ldots, \lambda_n \in F$.

linearly dependent
dependence witness
linearly independent
spans
spanning set

Definition 5.1.3. Let F be a field, V be a vector space over F, and X a subset of V. We say that X is ***linearly dependent*** over F, (l.d. for short) if there are $n \geq 1$, $x_1, \ldots, x_n \in X$, and $\lambda_1, \ldots, \lambda_n \in F^*$ such that

$$\lambda_1 x_1 + \cdots + \lambda_n x_n = 0, \tag{5.2}$$

i.e. a linear combination with non-zero coefficients, equal to 0. We call the Equation 5.2 a ***witness to the dependency*** of X, and say that the subset $\{x_1, \ldots, x_n\}$ witnesses that dependency.
We say that X is ***linearly independent*** over F (l.i. for short), if it is not linearly dependent over F.

To say that X is linearly independent means that there is no witness to dependence, in other words, that whenever we have

$$\lambda_1 x_1 + \cdots + \lambda_n x_n = 0$$

where $x_1, \ldots, x_n \in X$, and $\lambda_1, \ldots, \lambda_n \in F$, then all of the λ_i must be outside F^*, i.e. all of the λ_i must be equal to 0. Otherwise, we would drop those $\lambda_i = 0$, keeping those $\lambda_i \neq 0$, to get a witness to dependence. We can write this as an implication For $x_1, \ldots, x_n \in X$, and $\lambda_1, \ldots, \lambda_n \in F$,

$$(\lambda_1 x_1 + \cdots + \lambda_n x_n = 0) \Rightarrow (\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0.$$

Definition 5.1.4. Let V be a v.s. over F and $X \subseteq V$. We define $\mathrm{Span}(X)$ as the set of all linear combinations of X over F. We say that X ***spans*** V if $\mathrm{Span}(X) = V$. We also say that X is a ***spanning set*** of V.

Lemma 5.1.2 *Let V be a v.s. over F and $X \subseteq V$. Then* $\mathrm{Span}(X)$ *is a subspace of V.*

Proposition 5.1.3 *Let V be a v.s. over F, $X \subseteq Y \subseteq V$.*

1. *If Y is l.i. then X is l.i.*
2. $\mathrm{Span}(X) \leq \mathrm{Span}(Y)$.
3. *If X spans V then Y spans V.*

Definition 5.1.5. Let V be a v.s. over F, and $B \subseteq V$. We say that B is a basis for V over F if B is both linearly independent over F, and a spanning set for V over F.

5.2 Zorn's Lemma

partially ordered set
poset
total order
chain
maximum
largest
maximal

02/12/19 *Zorn's Lemma* is often called the algebraist's version of the axiom of choice. The reasons are that it is equivalent to the axiom of choice, and it is the most version of the axiom of choice used in algebra.

Definition 5.2.1. A *partially ordered set*, usually called *poset*, consists of a set A and a binary relation, generically denoted $\leq$, that is

- Reflexive
- Anti-symmetric
- Transitive
  When the order is *total*, we call it a *chain*.

Definition 5.2.2. Let $(A, \leq)$ be a poset. An element $t \in A$ is called *maximum* or *largest* if $(\forall a \in A)(a \leq t)$. An elemetnt $m \in A$ is called *maximal* if there is no $a \in A$ such that $m < a$. This is equivalent to saying

$$m \leq a \Rightarrow m = a.$$

Notes 5.2.1.

1. Maximal elements, when they exist, are not necessarily unique.
2. Due to anti-symmetry, when a maximum element exists, it is unique.
3. A poset does not need to have a maximum element or maximal elements.
4. If A has maximum element is maximal t, then t is maximal, and it is the only maximal element of A.
5. Any subset C of a poset $(A, \leq)$ is itself a poset with the same relation, restricted to C. When C is totally ordered, we say C is a chain in A.

As mentioned earlier, Zorn's Lemma is equivalent to the axiom of choice, and we will take it as an axiom.

Axiom 5.2.1 [Zorn's Lemma] *Let $(A, \leq)$ be a non-empty poset. If every non-empty chain in A has and upper bound in A, then A has a maximal element.*

An easy application of Zorn's Lemma shows the following proposition.

Proposition 5.2.1 *A PID D has a maximal ideal.*

Now back to vector spaces. In order to prove that any vector space V has a basis, we will prove a more general and more useful statement.

Lemma 5.2.2 *Let X be a l.i. set in a v.s. V. For $y \notin X$,*

$$X \cup \{y\} \quad \textit{is l.i. iff} \quad y \notin \mathrm{Span}(X).$$

Theorem 5.2.3 *Let V be a v.s. over F, $I \subseteq T \subseteq V$, such that I is l.i. and T spans V. There is $B \subset V$ such that $I \subseteq B \subseteq T$ and B is both l.i. and spanning set, i.e. a basis of V.*

Proof. ■

Corollary 5.2.4 *Let V be a v.s. over F. Then V has a basis.*

Proof. Apply Theorem 5.2.3 to the l.i. set $\emptyset$ and the spanning set V. ■

5.3 Bases and Dimension

02/13/19

Proposition 5.3.1 *Let B be a basis for v.s. V*

1. *B is maximal l.i. set.*
2. *B is minimal spanning set.*

Proof. This follows immediately from Lemma 5.2.2. ■

The next main theorem states that any two bases for a v.s. have the same cardinality. We need some lemmas before we can prove it.

The first lemma is a purely set theoretic result. It deals with the cardinality of the union of a family of finite sets.

Lemma 5.3.2 *Let $(X_i | i \in I)$ be a family of finite sets, and* $X = \bigcup_{i \in I} X_i$.

1. *If the indexing set I is finite then X is finite.*
2. *If the indexing set I is infinite then $|X| \leq |I|$.*

Lemma 5.3.3 [Exchange Lemma] *Let V be a v.s. over F, and B_1, B_2 bases for V. For every $x \in B_1$ there is a $y \in B_2$ such that $(B_1 - \{x\}) \cup \{y\}$ is a basis for V.*

Proof. Let $x \in B_1$. If $x \in B_2$, just take $y = x$. Otherwise, if $x \notin B_2$, let $W = \text{Span}(B_1 - \{x\})$. We claim that $W \subsetneqq V$, for otherwise we would get $x \in \text{Span}(B_1 - \{x\})$, and by Lemma 5.2.2 we would get B_1 is l.d. Since W is a proper subspace and B_2 is a spanning set, we cannot have $B_2 \subseteq W$, so there is $y \in B_2$ such that $y \notin W$. By Lemma 5.2.2, $B_3 := (B_1 - \{x\}) \cup \{y\}$ is l.i. Now, $B_3 \cup \{x\} = B_1 \cup \{y\}$, which is l.d. since B_1 is a spanning set. It follows that $x \in \text{Span}(B_3)$, so $B_1 \subseteq \text{Span}(B_3)$ and B_3 is also a spanning set. ■

The Exchange Lemma appears as one of the defining axioms for what are called *Matroids*. Matroids are an abstraction of the concept of linear independence.

Theorem 5.3.4 *Let V be a v.s. over F. If B_1 and B_2 are bases for V over F, then $|B_1| = |B_2|$.*

Proof. If B_1 and B_2 are both finite, then repeated application of the Exchange Lemma, shows that we can entirely replace all elements of B_1 with elements of B_2. Hence, $|B_1| \leq |B_2|$. Now, reverse the roles of B_1 and B_2 to get the other inequality.

finite dimensional
almost all zero

Assume now that at least one of the two bases is infinite. Each $y \in B_2$ is a l.c. of elements of B_1, so there is a finite set $X_y \subseteq_f B_1$, such that $y \in \mathrm{Span}(X_y)$. Let $X = \bigcup_{y \in B_2} X_y$. We have $X \subseteq B_1$, and $B_2 \subseteq \mathrm{Span}(X)$. Since B_2 is a spanning set, we get that X is also a spanning set, and by Proposition 5.3.1.2, we must have $X = B_1$. On the other hand, from Lemma 5.3.2 if we had B_2 finite, that would make X finite, so B_2 has to be infinite, and $|B_1| = |X| \leq |B_2$. Now reverse the roles of B_1 and B_2 to get the other inequality. ■

Definition 5.3.1. Let V be a v.s. over F. We denote by $\dim_F(V)$ the cardinality of any basis for V. We say that V is *finite dimensional* over F if $\dim_F(V)$ is finite, i.e. if V has a finite basis.

Examples 5.3.1.

1. $F[x]$ is a countably infinite dimensional v.s. over F, with basis $\{1, x, x^2, \dots\}$, the set of all powers of x.
2. $F_n[x] = \{f \in F[x] | \deg(f) < n\}$ is an n-dimensional v.s. over F, with basis $\{1, x, x^2, \dots, x^{n-1}\}$.
3. $\dim_F(F) = 1$, with basis $\{1\}$.
4. $\dim_F(\{0\}) = 0$, with empty basis.

Example 5.3.2. $\{1, \sqrt{2}\}$ is a basis for $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$.

02/15/19

Lemma 5.3.5 *Let V be a v.s. and B a basis for V. For any $x \in V$ there is a unique l.c. of elements in B, equal to x.*

The uniqueness in Lemma 5.3.5 is to be understood up to order, and with the understanding that any basis vector not showing explicitly in the l.c., is present with coefficient 0. In fact, the expression

$$\sum_{v \in B} \lambda_v v$$

makes sense when B is infinite, provided that the coefficients $(\lambda_v \mid v \in B)$ are *almost all zero*, i.e. all but finitely many of them are zero.

5.4 Linear Transformations

linear transformation

Homomorphisms of vector spaces are called linear transformations.

Definition 5.4.1. Let V, W be vector spaces over the same field F. A function $\varphi : V \to W$ is called a *linear transformation* if it preserves the operations of V, i.e. addition and scalar multiplication. For any $v, v_1, v_2 \in V$ and $\lambda \in F$,

$$\varphi(v_1 + v_2) = \varphi(v_1) + \varphi(v_2) \qquad \text{and} \qquad \varphi(\lambda v) = \lambda\varphi(v).$$

Proposition 5.4.1 [Universal Mapping Property for Vector Spaces] *Let V, W be vector spaces over F, and B a basis for V. Given a function $\alpha : B \to W$, there is a unique l.t. $\widehat{\alpha} : V \to W$ such that for each $v \in B$, $\widehat{\alpha}(v) = \alpha(v)$.*

Proof. Each $x \in V$, can be uniquely written as

$$x = \sum_{v \in B} \lambda_v v$$

where almost all λ_v's are zero. Define

$$\widehat{\alpha}(x) := \sum_{v \in B} \lambda_v \alpha(v).$$

It is easy to check that $\widehat{\alpha}$ as just defined, has the required properties. ∎

Proposition 5.4.2 *Given the set up of Proposition 5.4.1, we have:*

1. *the set $\{\alpha(v) \mid v \in B\}$ is linearly independent in W iff $\widehat{\alpha}$ is injective.*
2. *the set $\{\alpha(v) \mid v \in B\}$ is a spanning set for W iff $\widehat{\alpha}$ is surjective.*

Exercise 5.4.1. This exercise refers to Proposition 5.4.1.

1. Complete the proof of Proposition 5.4.1.
2. Prove that the set $\{\alpha(v) \mid v \in B\}$ is linearly independent in W iff $\widehat{\alpha}$ is injective.

quotient space

3. Prove that the set $\{\alpha(v) \mid v \in B\}$ is a spanning set for W iff $\widehat{\alpha}$ is surjective.

The next theorem, in conjunction with Theorem 5.2.3, provide what is often called a "*complete classification*" of vector spaces over a field F. They describe what all vector spaces look like, and tell exactly when two vector spaces are the same, up to isomorphism.

Theorem 5.4.3 *Two vector spaces over a field F are isomorphic iff they have the same dimension.*

Proof. ($\Rightarrow$) Let $\varphi : V \to W$ be an isomorphism, and let B be a basis for V. By Proposition 5.4.2, the set $\{\varphi(v) \mid v \in B\}$ is a basis for W and it has the same cardinality as B.
($\Leftarrow$) Let V, W be vector spaces over F with the same dimension. Let B_1, B_2 be bases for V and W, respectively. Since $\dim(V) = \dim(W)$, there is a bijection $\alpha : B_1 \to B_2$. By Proposition 5.4.1 there is a l.t. $\widehat{\alpha} : V \to W$. Since $\{\alpha(v) \mid v \in B_1\} = B_2$, and B_2 is l.i. and spanning, by Proposition 5.4.2, $\widehat{\alpha}$ is injective and surjective. ∎

Exercise 5.4.2. Let W be a subspace of a vector space V.

1. Prove that $\dim(W) \leq \dim(V)$.

2. If V is finite dimensional and $\dim(W) = \dim(V)$ then $W = V$.

3. Show, with a counterexample, that part (2) does not hold without the finite dimensional hypothesis.

5.5 Quotient Space and Isomorphism Theorem

Definition 5.5.1. Given a vector space V over F, and a subspace W, we define the ***quotient space*** V/W as

$$V/W = \{v + W \mid v \in V\}$$

with operations defined via representatives, just like in the case of groups and rings.

$$(v_1 + W) + (v_2 + W) := (v_1 + v_2) + W \qquad \text{and} \qquad \lambda(v + W) := (\lambda v) + W,$$

for any $v, v_1, v_2 \in V$ and $\lambda \in F$.

It is a straightforward exercise to show that V/W with the given operations is a vector space over F

There are isomorphism theorems similar to those of groups and rings. In particular, the first isomorphism theorem for vector spaces can be phrased as follows:

Exercise 5.5.1. Let V be a vector space over F, and W a subspace of V. Let B_1 be a basis for W and B a basis for V such that $B_1 \subseteq B$. Prove that the set

$$\{v + W \mid v \in B - B_1\}$$

is a basis for the quotient space V/W.

Theorem 5.5.1 *Let V, W be vector spaces over F, and $\varphi : V \to W$ a linear transformation. Then*

1. $\mathrm{Im}(\varphi)$ *is a subspace of* W,
2. $\ker(\varphi)$ *is a subspace of* V,
3. $$V/\ker(\varphi) \approx \mathrm{Im}(\varphi)$$
4. *When V is finite dimensional,*
$$\dim_F(V) = \dim_F(\ker(\varphi)) + \dim_F(\mathrm{Im}(\varphi)).$$

Theorem 5.5.1.4 is usually called the *Nullity-Rank Theorem* in linear algebra courses.

When V is intinite-dimensional, the formula in Theorem 5.5.1.4 is still valid, with the right hand side being addition of cardinals.

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