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The Number of Holes in the Union of Translates of a Convex Set in Three Dimensions

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Abstract We show that the union of n translates of a convex body in $\mathbb{R}^3$ can have $\Theta(n^3)$ holes in the worst case, where a *hole* in a set X is a connected component of $\mathbb{R}^3 \setminus X$. This refutes a 20-year-old conjecture. As a consequence, we also obtain improved lower bounds on the complexity of motion planning problems and of Voronoi diagrams with convex distance functions.

Keywords Union complexity · Convex sets · Motion planning

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1 Introduction

From path planning in robotics [18] to the design of epsilon-nets [7] to analyzing vulnerabilities in networks [2], a variety of combinatorial and algorithmic problems in computational geometry involve understanding the complexity of the union of n elementary objects. An abundant literature studies how this union complexity depends on the geometry of the objects, and we refer the interested reader to the survey of Agarwal et al. [1]. In the plane, two important types of conditions were shown to imply near-linear union complexity: restrictions on the number of boundary intersections [12,19] and fatness [8,17]. In three dimensions, the former are less relevant as they do not apply to important examples such as motion planning problems. Whether fatness implies low union complexity in $\mathbb{R}^3$ has been identified as an important open problem in the area [15, Prob. 4]; to quote Agarwal et al. [1, Sect. 3.1, §2],

> *A prevailing conjecture is that the maximum complexity of the union of such fat objects is indeed at most nearly quadratic. Such a bound has however proved quite elusive to obtain for general fat objects, and this has been recognized as one of the major open problems in computational geometry.*

A weaker version of this conjecture asserts that the number of holes in the union of n translates of a fixed convex body in $\mathbb{R}^3$ is at most nearly quadratic in n. By a *hole* in a subset $X \subseteq \mathbb{R}^d$ we mean a connected component of $\mathbb{R}^d \setminus X$. This is indeed a weakening since the number of holes is a lower bound on the complexity and a family of translates can be made quite fat by applying a suitable affine transformation. Evidence in support of the weaker conjecture is that in two dimensions, the union of n translates has at most a linear number of holes [12], and in three dimensions, the union of n translates of a convex polytope with k facets has at most $O(kn^2)$ holes [3], so the number of holes grows only quadratically for any fixed convex polytope.

Remarkably, we refute the conjecture even in this weaker form. We construct a convex body in $\mathbb{R}^3$ that has, for any n, a family of n translates with $\Theta(n^3)$ holes in its union. This matches the upper bound for families of arbitrary convex bodies in $\mathbb{R}^3$ [14].

Theorem 1 *The maximum number of holes in the union of n translates of a compact, convex body in* $\mathbb{R}^3$ *is* $\Theta(n^3)$.

We start with a warm-up example illustrating the idea behind our construction (Sect. 2). We then construct a polytope K_m, tailored to the value of m considered and with $\Theta(m^2)$ faces; we give a family of $3m$ translates of K_m (in Sect. 3) whose union has $\Theta(m^3)$ holes. The final step of our construction is to turn the family of polytopes (K_m) into a limiting "universal" convex body K that, for any m, admits $3m$ translates whose union has $\Theta(m^3)$ holes. We prove this formally using arguments from algebraic topology, developed in Sect. 5.

The convex body K that we construct is not centrally symmetric and this is essential for the proof of Theorem 1. Existence of a centrally symmetric convex set with a cubic number of holes in the union of its translates is an interesting open problem (as it would, for instance, generalize Corollary 2 from distance functions to norms, see below).

1.1 Further Consequences

We conclude this introduction with two examples of problems in computational geometry whose underlying structure involves a union of translates of a convex body, and on which Theorem 1 casts some new light.

A *motion-planning problem* asks whether an object, typically in $\mathbb{R}^2$ or $\mathbb{R}^3$, can move from an initial position to a final position by a sequence of elementary motions while remaining disjoint from a set of obstacles (and to compute such a motion when it exists). This amounts to asking whether two given points lie in the same connected component of the *free space*; that is, the set of positions of the object where it intersects no obstacle. When the motions are restricted to translations, the free space can be obtained by taking the complement of the union of the "expansion" (formally: the *Minkowski sum*) of every obstacle by the reflection of the object through the origin. In the simplest case the mobile object is convex, the obstacles consist of n points and the free space is the complement of the union of n translates of a convex body; Theorem 1 implies that already in this case the free space can have large complexity:

Corollary 1 *There exists a set P of n point obstacles and a convex body K in $\mathbb{R}^3$ such that the free space for moving K by translations while avoiding P has $\Theta(n^3)$ connected components.*

The *Voronoi diagram* of a family P of points $p_1, p_2, \ldots, p_n$, called *sites*, in a metric space X is the partition of X according to the closest p_i. A subset $Q \subseteq P$ *defines a face* of the diagram if there exists some point $x \in X$ at equal distance from all sites of Q, and strictly further away from all sites in $P \setminus Q$. A case of interest is when X is $\mathbb{R}^d$ equipped with a *convex distance function* d_U defined by a convex unit ball U (in general, d_U is not a metric, as U needs not be centrally symmetric). A *face* of the Voronoi diagram with respect to d_U is defined to be a connected component of the set

$$\{x : d(x, q) \leq d(x, p), \text{ for all } q \in Q, p \in P\}$$

for some $Q \subset P$ defining a face; the *complexity* of a Voronoi diagram is measured by the number of its faces of all dimensions. In $\mathbb{R}^2$, the complexity of the Voronoi diagram of n point sites with respect to d_U is $O(n)$, independent of the choice of U [5,9].

The state of knowledge is less satisfactory in three dimensions where we know near-quadratic complexity bounds for the Voronoi diagram with respect to d_U if U is a constant complexity polytope [13], and that by making U sufficiently complicated one can have four point sites define arbitrarily many Voronoi vertices (that is, isolated points at equal distance from these four sites) [11]. If we grow equal-radii balls (for d_U) centered at each of the p_i simultaneously, every hole in the union of these balls must contain a Voronoi vertex. Theorem 1 therefore implies that one can also find $\Omega(n^3)$ different quadruples of point sites defining Voronoi vertices.

Corollary 2 *There exist a convex distance function d_U and a set P of n points in $\mathbb{R}^3$ such that $\Omega(n^3)$ different quadruples of P define a Voronoi vertex in the Voronoi diagram of P with respect to d_U.*

1.2 Notation

For an integer n we let $[n]$ denote the set $\{1, \dots, n\}$. If A and B are two subsets of $\mathbb{R}^d$ we denote by $A + B$ and $A - B$ the *Minkowski sums*

$$A + B = \{a + b : a \in A, b \in B\} \quad \text{and} \quad A - B = \{a - b : a \in A, b \in B\},$$

and by $\mathrm{conv}(A)$ the *convex hull* of A. We refer to coordinates in three space by x, y, z. We refer to the positive z-direction as "upward," and to the positive y-direction as "forward." For an object $A \subset \mathbb{R}^3$, the "front" boundary of A is the upper y-envelope. Similarly, the "back" boundary is the lower y-envelope.

1.3 Remark on Figures

The reader should take heed that in several places we give explicit coordinates for a construction and provide a figure, where the figure depicts the qualitative geometric features of interest using different coordinates. Using the coordinates chosen for convenience of computation would have resulted in figures where features are too small to see.

2 Many Holes Touching a Single Facet

We first introduce the key idea of our constructions with a simpler goal: constructing $2m + 1$ translates of a convex polytope with $\Omega(m^2)$ holes in the union, all incident to a common facet.

Let K be the polytope depicted in Fig. 1a, the convex hull of the following seven points:

$$a = (0, 0, 0), \quad b = (1, 0, 0), \quad c = (0, 0, -1), \quad d = (1, 0, -1),$$
$$e = (1/2, 1/2, 1/2), \quad f = (0, 1, 0), \quad \text{and} \quad g = (1, 1, 0).$$

We fix an integer $m > 0$ and let $C = K$ be the trivial translate of K. Let F denote the facet of C with vertices a, b, c, d. We then pick m translates of K, denoted $B_1, B_2, \dots, B_m$, whose top-most vertices (corresponding to vertex e) are placed regularly along the edge ab of F. The top part of the intersection $B_j \cap F$ is a triangular region, shown in blue in Fig. 1e. The union $\bigcup_{j=1}^{m} B_j$ bounds $m - 1$ regions below the edge ab of F. Let ℓ denote the height of one such region; that is the distance between $B_1 \cap B_2$ and the edge ab of F. Let c' denote the point on the segment ac of F at distance ℓ from a (see again Fig. 1e). We next pick m translates of K, denoted $A_1, A_2, \dots, A_m$, whose vertices corresponding to f are placed regularly along the segment ac'.

The intersections $F \cap B_j$ and $F \cap A_i$, for $i, j \in [m]$, form a grid in F with $m(m-1)$ holes on F. Since two consecutive A_i leave only a narrow tunnel incident to F, and each B_j entirely cuts this tunnel, each of the holes on F is indeed on the boundary of a distinct hole in the union of all translates in $\mathbb{R}^3$.

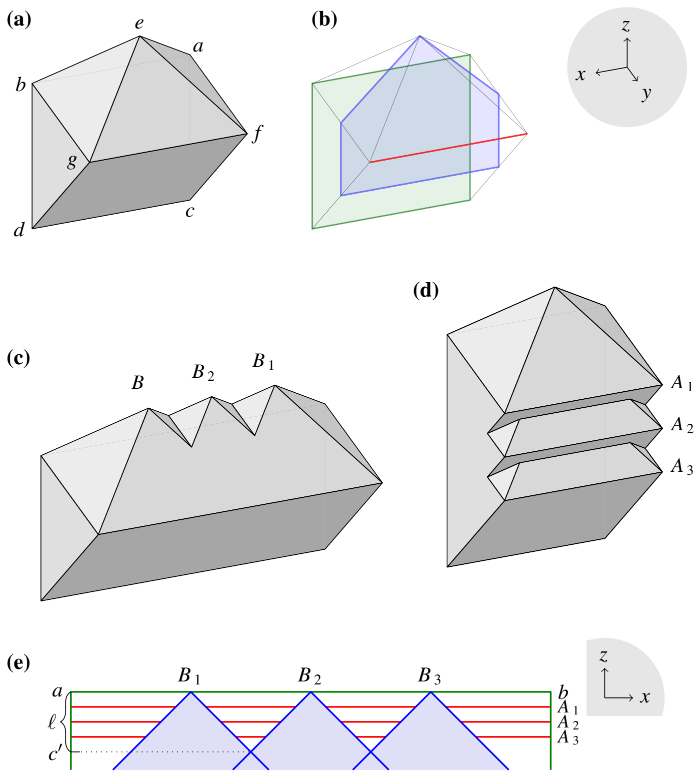

Fig. 1 **a** The convex polytope K for the construction of Sect. 2. **b** Three important (x, z) cross sections of K. The cross section in the back (*green*) is the facet F that touches many holes, the one in the center (*blue*) is used to produce vertical cones, and the one in the front (*red*) is used to produce horizontal edges. **c** Union of the translates $B_1, B_2, \ldots, B_m$. **d** Union of the translates $A_1, A_2, \ldots, A_m$. (The spacing between the translates is exaggerated for clarity.) **e** The grid formed on the facet F by horizontal segments and vertical cones

Claim The union $\bigcup_{i=1}^{m} A_i \cup \bigcup_{j=1}^{m} B_j \cup C$ has $\Omega(m^2)$ holes, all touching the facet F of C.

Since this is only a warm-up example, we do not include a formal proof of the claim.

3 The Construction of K_m

The construction of Sect. 2 uses three features of K: a portion of a cone with apex e, a portion of a prism with edge fg, and a facet F. We combined the A_i's and the B_j's in a grid-like structure that created $\Theta(m^2)$ local minima in the y-direction on the front boundary of the union $\bigcup A_i \cup \bigcup B_j$. Each of these minima is the bottom of a "pit," and C acts as a "lid" to turn each pit into a separate hole. The proof of Theorem 1 is based on a similar principle. As a first step, we first construct, for some value $m > 0$, a convex polytope K_m depending on m and a family of $3m$ translates with $\Theta(m^3)$ holes. The set K_m consists of two parts, which we refer to as Front and Back.

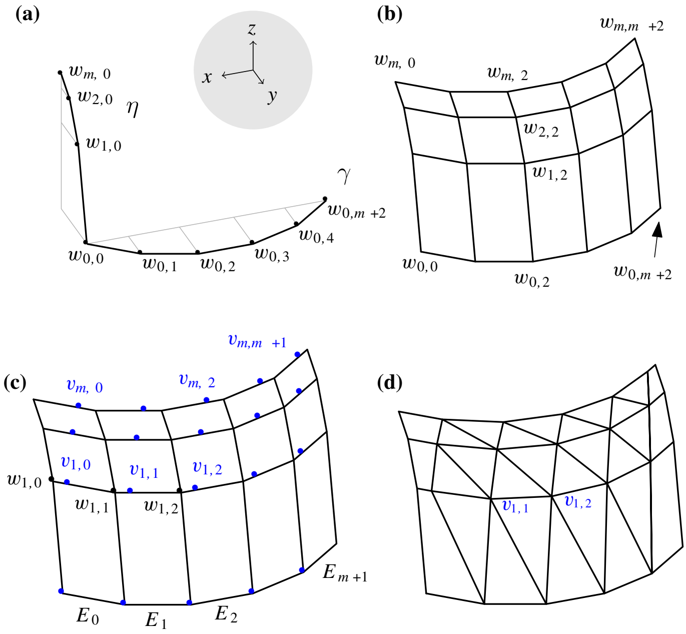

Fig. 2 Design of FRONT. **a** The auxiliary paths η and γ. **b** The surface $\eta+\gamma$ defining the "grid" of vertices $w_{j,k}$. **c** The (*blue*) points $v_{j,k}$. **d** The front boundary of FRONT

3.1 The Auxiliary Paths

To construct our polytope K_m we use two auxiliary polygonal paths η and γ. They both start at the origin 0. The path η has m edges, lies in the (y,z)-plane, and is convex[1] in both directions $(0,-1,0)$ and $(0,0,-1)$. The path γ has $m+2$ edges, lies in the (x,y)-plane, and is convex in direction $(0,-1,0)$. We denote the jth vertex of η by $w_{j,0}$ and the kth vertex of γ by $w_{0,k}$, see Fig. 2a.

3.2 The Front Part

We define a "grid" of $(m+1)\times(m+3)$ points, which are the vertices of the polyhedral surface $\eta+\gamma$, see Fig. 2b, by putting

$$w_{j,k} = w_{j,0} + w_{0,k}$$

for $j \in \{0,1,\ldots,m\}$ and $k \in \{0,1,\ldots,m+2\}$. We then add a point $v_{j,k}$ on the edge $w_{j,k}w_{j,k+1}$ as follows; see Fig. 2c:

$$v_{j,k} = \left(1 - \tfrac{j}{m+1}\right) w_{j,k} + \tfrac{j}{m+1} w_{j,k+1}$$

[1] We say a path π is convex in a *direction* u if the orthogonal projection of π onto $u^{\perp}$ is injective and the set $\pi + \mathbb{R}^{+}u$ is convex.

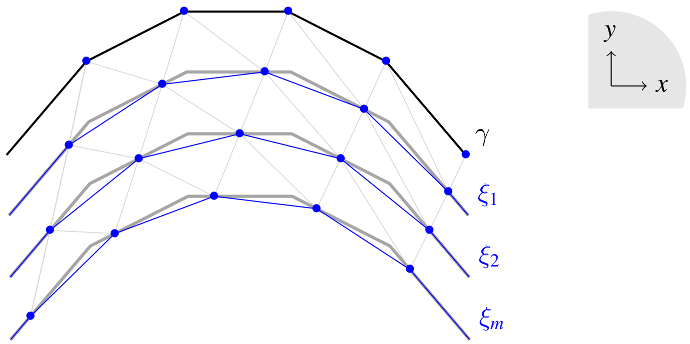

Fig. 3 View of the paths ξ_j from above (in *blue*, with the paths γ_j represented in *grey*)

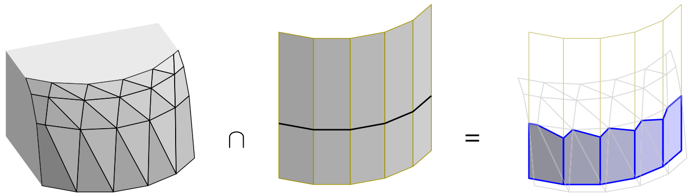

Fig. 4 Intersection of vertical panels through the path γ_1 with FRONT

for $j \in \{0, 1, \ldots, m\}$ and $k \in \{0, 1, \ldots, m+1\}$. For $j \in \{0, \ldots, m\}$ we define two polygonal paths

$$\gamma_j = w_{j,0} + \gamma = w_{j,0}w_{j,1}w_{j,2}\ldots w_{j,m+1}w_{j,m+2} \text{ and}$$
$$\xi_j = w_{j,0}v_{j,0}v_{j,1}v_{j,2}v_{j,3}\ldots v_{j,m}v_{j,m+1}w_{j,m+2}.$$

Note that $\xi_0 = \gamma_0 = \gamma$, that the paths γ_j are simply translates of γ, that the path ξ_j lies entirely in the convex region $\gamma_j + (0, \mathbb{R}^-, 0)$, and that the vertices of ξ_j lie on γ_j; see Fig. 3.

The front part FRONT of K_m is the convex hull of the paths ξ_j; see Fig. 2d:

$$\text{FRONT} = \text{conv}(\xi_0 \cup \xi_1 \cup \cdots \cup \xi_m).$$

Observe that the paths η and γ can be chosen so that, for each $j \in [m]$, the path ξ_j lies entirely on the front boundary as well as on the upper boundary of FRONT.

Let $E_{j,k}$ denote the edge $w_{j,k}w_{j,k+1}$ and let $E_k = E_{0,k}$. Consider the point $v_{j,k}$ on $E_{j,k}$. Since ξ_j lies entirely on the upper boundary of FRONT, no part of FRONT appears above $E_{j,k}$, and since $v_{j,k}$ is the only point where the segment $E_{j,k}$ intersects FRONT for $j, k \in [m]$, the vertical plane containing $E_{j,k}$ intersects FRONT in a downward extending cone with apex $v_{j,k}$; see Fig. 4. Note that this fails for $k = 0$ and for $k = m+1$.

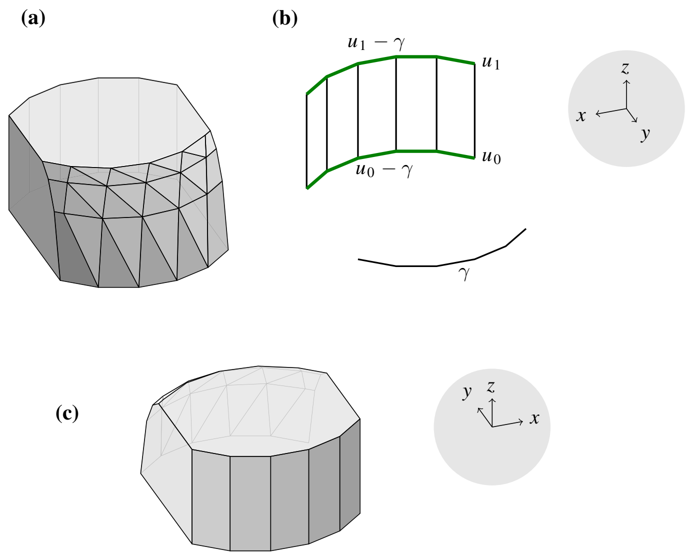

Fig. 5 Design of BACK. **a** The polytope K_m. **b** The back boundary of K_m, formed by extruding $-\gamma$ vertically. **c** K_m as viewed from behind

3.3 The Back Part

We now define two vectors, $u_1 = w_{m,m+2} - (0, t, 0)$, where t is some positive real number, and u_0, the orthogonal projection of u_1 on the (x, y)-plane. We define BACK as $u_0 u_1 - \gamma$, that is, the Minkowski sum of the segment $u_0 u_1$ and the reflection of γ with respect to the origin; see Fig. 5b. The value of t will be chosen so that FRONT and BACK have disjoint convex hulls.

By construction, BACK consists of $m + 2$ rectangles orthogonal to the (x, y)-plane, namely the rectangles $u_0 u_1 - E_k$ formed as the Minkowski sum of a horizontal and a vertical segment, for $k \in \{0, \ldots, m + 1\}$. The top and bottom edges of each rectangle are $u_1 - E_k$ and $u_0 - E_k$ respectively.

3.4 The Polytope K_m and Its Translates

We now let $K_m = \text{conv}(\text{FRONT} \cup \text{BACK})$, see Fig. 5a, and define three families of its translates. First, for $k \in [m]$, we define a translate C_k such that the edge $u_1 - E_k$ of C_k coincides with the edge E_k of K_m. Formally, we have

$$C_k = K_m + c_k, \quad \text{where} \quad c_k = w_{0,k} - (u_1 - w_{0,k+1});$$

see Fig. 6c. The vertical facet formed by $u_1 - E_k$ and $u_0 - E_k$ of C_k will be incident to a quadratic number of holes, playing the same role as the facet F in the previous section. Let us denote this facet as R_k.

Next, for $j \in [m]$, we define a translate B_j of K_m as follows:

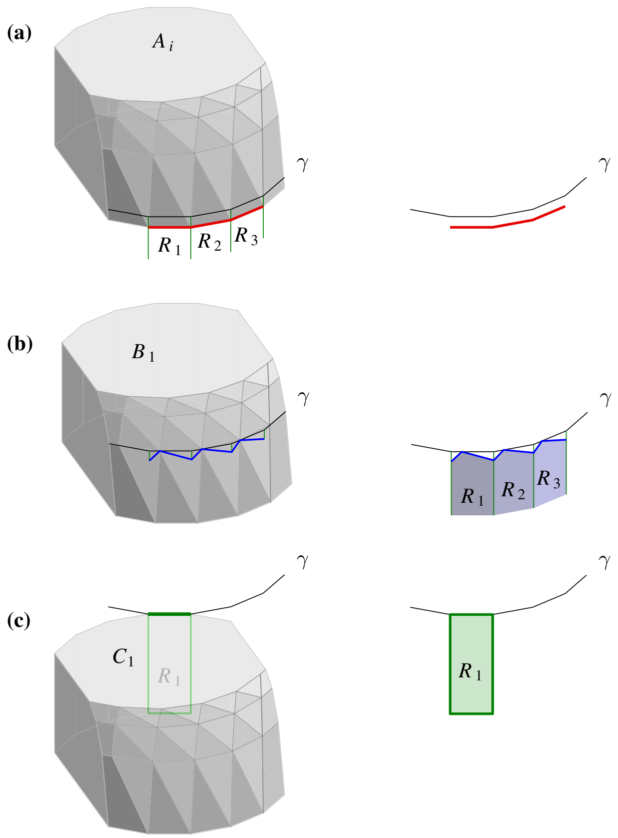

Fig. 6 The intersection of translate A_i, B_1, and C_1 with the facets $R_1, \dots, R_m$

$$B_j = K_m + b_j, \quad \text{where} \quad b_j = -w_{j,0}.$$

In other words, the path ξ_j of B_j lies in the (x, y)-plane and the vertex $v_{j,k}$ of B_j lies on E_k; see Fig. 6b.

Consider now the edge E_k, for $k \in [m]$. For each $j \in [m]$, the edge E_k contains the translated vertex $v_{j,k}$ of B_j. By the argument above, the intersection of B_j with the facet R_k is a vertical cone with apex

$$v_{j,k} + b_j = \big(1 - \tfrac{j}{m+1}\big) w_{0,k} + \big(\tfrac{j}{m+1}\big) w_{0,k+1}.$$

These apices are regularly spaced along E_k, and we obtain a configuration similar to Fig. 1. In other words, the union $\bigcup_{j \in [m]} B_j$ bounds $m - 1$ triangular regions below the edge E_k on the facet R_k of C_k.

We now pick a sufficiently small number $\varepsilon > 0$ and define our final family of translates. For $i \in [m]$, let

$$A_i = K_m + a_i, \quad \text{where } a_i = \big(0, 0, -\tfrac{i}{m}\varepsilon\big).$$

The translates A_i are defined by translating K_m vertically such that for every $i \in [m]$ and $k \in [m]$, the edge E_k of A_i appears as a horizontal edge on R_k, cutting each of the $m-1$ triangular regions; see Fig. 6a.

3.5 The Family $\mathcal{F}$

We now finally set $\mathcal{F} = \{A_1, \dots, A_m, B_1, \dots, B_m, C_1, \dots, C_m\}$. The union of the family $\mathcal{F}$ has at least $m^2(m-1)$ holes: For $k \in [m]$, we consider the facet R_k of C_k. On this facet, the union of the B_j and A_i forms a grid with $m(m-1)$ holes. As in Sect. 2, we argue that each of these holes is incident to a distinct component of the complement of the union of all the translates.

We will not give a formal proof of this fact, since we will present an even stronger construction in Sect. 4, and we will include a formal, algebraic argument for the correctness of that construction.

3.6 Explicit Coordinates for K_m

The reader not satisfied with the qualitative description of $\mathcal{F}$ may enjoy verifying that the following coordinates satisfy the properties we needed for our construction:

$$\begin{aligned} w_{j,0} &= (0, -3j, 1-2^{-j}) && \text{for } 0 \le j \le m, \\ w_{0,k} &= (\cos\theta_k - 1, \sin\theta_k, 0) && \text{for } \theta_k = \tfrac{\pi}{3}\big(\tfrac{k-1}{m} + 1\big), 1 \le k \le m+1, \\ w_{0,m+2} &= (-2, 0, 0), \\ u_1 &= (-2, -3m-3, 1-2^{-m}), \\ u_0 &= (-2, -3m-3, 0). \end{aligned}$$

4 Constructing a Universal Convex Body

The family of translates constructed in Sect. 3 uses a convex polytope K_m that depends on m. In this section we construct a single convex body K that allows the formation of families of n translates of K, for arbitrarily large n, with a cubic number of holes. (Note that the *position* of the translates in the family will depend on n.)

4.1 The Convex Body K

The finite polygonal paths γ and η are replaced by infinite polygonal paths. We must also redefine the vertices $v_{j,k}$, but the rest of the construction remains largely the same as before. (Figure 7 illustrates the construction.)

Using $\zeta_s = \sum_{t=1}^{\infty} t^{-s}$ for $s \in \{2, 3\}$, we define vertices as follows:

$$\begin{aligned} w_{0,0} &= 0, \\ w_{j,0} &= w_{j-1,0} + (0, -j^{-2}, j^{-3}) && \text{for } j \in \{1, 2, \dots\}, \\ w_{\infty,0} &= \lim_{j\to\infty} w_{j,0} = (0, -\zeta_2, \zeta_3), \end{aligned}$$

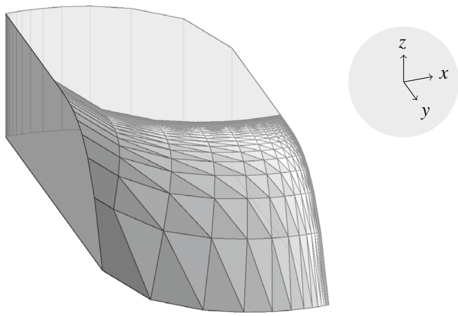

Fig. 7 A convex body K, not depending on n, with translates forming $\Theta(n^3)$ holes

$$
\begin{aligned}
w_{0,k} &= w_{0,k-1} + (k^{-2}, k^{-3}, 0) && \text{for } k \in \{1, 2, \dots\},\\
w_{0,\infty} &= \lim_{k\to\infty} w_{0,k} = (\zeta_2, \zeta_3, 0),\\
w_{j,k} &= w_{j,0} + w_{0,k} && \text{for } j, k \in \{1, 2, \dots, \infty\},\\
v_{j,k} &= \tfrac{1}{(j+1)^3} w_{j,k} + \tfrac{(j+1)^3-1}{(j+1)^3} w_{j,k+1} && \text{for } j, k \in \{0, 1, \dots\},\\
u_1 &= (\zeta_2, -2, \zeta_3),\\
u_0 &= (\zeta_2, -2, 0).
\end{aligned}
$$

Again we let E_k denote the edge $w_{0,k}w_{0,k+1}$, for $k \in \{0, 1, \dots\}$. The vertex $v_{j,k}$ lies on the edge $w_{j,k}w_{j,k+1} = E_k + w_{j,0}$. We define the convex paths

$$
\begin{aligned}
\gamma &= w_{0,0}w_{0,1}w_{0,2}\dots w_{0,\infty},\\
\eta &= w_{0,0}w_{1,0}w_{2,0}\dots w_{\infty,0},\\
\xi_j &= w_{j,0}v_{j,0}v_{j,1}v_{j,2}v_{j,3}\dots w_{j,\infty}.
\end{aligned}
$$

Note that $\xi_0 = \gamma$. For $j \in \{1, 2, \dots\}$, we set $\gamma_j = w_{j,0} + \gamma$. The path ξ_∞ is equal to $\gamma_\infty = \gamma + w_{\infty,0}$; note that $\lim_{j\to\infty} v_{j,k} = w_{\infty,k+1}$.

The front part of K is the convex hull of the paths ξ_j:

$$\text{Front} = \operatorname{conv}(\xi_0 \cup \xi_1 \cup \xi_2 \cup \dots \cup \xi_\infty).$$

The back part Back of K is the Minkowski sum of u_1u_0 and $-\gamma$. By construction, it is the union of the rectangles $u_1u_0 - E_k$, for $k \in \{0, 1, 2, \dots\}$. The top and bottom edges of each rectangle are $u_1 - E_k$ and $u_0 - E_k$.

Finally, we define $K = \operatorname{conv}(\text{Front} \cup \text{Back})$, concluding the description of our convex body K. This body has the following property.

Lemma 1 *The polygonal path ξ_j lies entirely on the front boundary of K.*

Proof We will show that any point in ξ_j lies outside the convex hull of the regions $\xi_{j'} + (0, \mathbb{R}^-, 0)$, for $j' \neq j$. Fix $j \in \mathbb{N}$, and let Γ be the convex hull of the regions $\gamma_{j'} + (0, \mathbb{R}^-, 0)$, for $j' \neq j$. Since the path $\xi_{j'}$ lies in the convex region $\gamma_{j'} + (0, \mathbb{R}^-, 0)$, it suffices to show that ξ_j lies outside Γ. Since the γ_j are just translates of γ, the body Γ is easy to describe. In particular, between the (x, y)-parallel planes containing γ_{j-1} and γ_{j+1}, its front boundary is formed by rectangles that are the convex hull of $E_k + w_{j-1,0}$ and $E_k + w_{j+1,0}$. Let Π be the (x, y)-parallel plane containing ξ_j and γ_j. The front boundary of $\Gamma \cap \Pi$ is again a translate of γ. We first compute this translate of γ, by finding the intersection point p of the segment $w_{j-1,0}w_{j+1,0}$ with Π.

Let $p' = p - w_{j-1,0}$. This makes p' the point at height $1/j^3$ on the line through the line through the origin and $w_{j+1,0} - w_{j-1,0}$. Since

$$\begin{aligned} w_{j+1,0} - w_{j-1,0} &= \Big(0, -\frac{1}{j^2} - \frac{1}{(j+1)^2}, \frac{1}{j^3} + \frac{1}{(j+1)^3}\Big) \\ &= \Big(0, \frac{-j^2 - (j+1)^2}{j^2(j+1)^2}, \frac{j^3 + (j+1)^3}{j^3(j+1)^3}\Big), \end{aligned}$$

this gives

$$p' = \Big(0, \frac{-(j+1)(j^2 + (j+1)^2)}{j^2(j^3 + (j+1)^3)}, \frac{1}{j^3}\Big).$$

Since $p = p' + w_{j-1,0}$ and $w_{j,0} - w_{j-1,0} = (0, -j^{-2}, j^{-3})$, we have

$$p = w_{j,0} - \Big(0, \frac{1}{j^3 + (j+1)^3}, 0\Big).$$

We have just computed the front boundary $\gamma + p$ of $\Gamma \cap \Pi$, and we want to show that the path ξ_j lies farther in front of this boundary. Since $\gamma + p$ and ξ_j are convex paths, it suffices to show that any vertex of $\gamma + p$ lies behind ξ_j. These vertices are the points $w_{0,k} + p$, for $k \in \{0, 1, 2, \dots\}$. That is, we will show $w_{0,k} + p \in \xi_j + (0, \mathbb{R}^-, 0)$.

We fix some $k \in \{1, 2, \dots\}$ and consider $w_{0,k} + p$. The line parallel to the y-axis through $w_{0,k} + p$ intersects the edge $v_{j,k-1}v_{j,k}$ of ξ_j in a point q. We need to show that $q^y \geq (w_{0,k} + p)^y$ (here and in the following, we use superscripts x, y, z to denote the coordinates of a point). It will be convenient to translate our coordinate system such that $w_{j,k}$ is the origin. This means that Π is the plane $z = 0$. Letting $J = (j+1)^3$, we have:

$$\begin{aligned} w_{0,k} + p - w_{j,k} &= w_{0,k} + p - (w_{j,0} + w_{0,k}) = \Big(0, \frac{-1}{j^3 + J}, 0\Big), \\ v_{j,k} - w_{j,k} &= \tfrac{1-J}{J} w_{j,k} + \tfrac{J-1}{J} w_{j,k-1} = \tfrac{J-1}{J}[w_{j,k+1} - w_{j,k}], \\ v_{j,k-1} - w_{j,k} &= \tfrac{1}{J} w_{j,k-1} - \tfrac{1}{J} w_{j,k} = \tfrac{1}{J}[w_{j,k-1} - w_{j,k}]. \end{aligned}$$

Now parameterize the segment $v_{j,k-1}v_{j,k}$ as $E(s)$, for $0 \leq s \leq 1$, using our new coordinate system:

$$E(s) = (1-s)\frac{J-1}{J}[w_{j,k+1} - w_{j,k}] + \frac{s}{J}[w_{j,k-1} - w_{j,k}].$$

The x- and y-coordinates of the point $E(s)$ are

$$E(s)^x = (1-s)\frac{J-1}{J}\frac{1}{(k+1)^2} - \frac{s}{J}\frac{1}{k^2},$$
$$E(s)^y = (1-s)\frac{J-1}{J} - \frac{1}{(k+1)^3} - \frac{s}{J}\frac{1}{k^3}.$$

We have $q - w_{j,k} = E(t)$, for the $t \in [0,1]$ where $E(t)^x = 0$. This condition is equivalent to $(1-t)(J-1)k^2 = t(k+1)^2$, and therefore

$$t = \frac{(J-1)k^2}{(k+1)^2 + (J-1)k^2},$$
$$E(t)^y = \frac{1-J}{J}\frac{1}{k(k+1)((k+1)^2+(J-1)k^2)} \geq \frac{1-J}{J}\frac{1}{2(J+3)},$$

where we used $k(k+1) \geq 2$, $k^2 \geq 1$, $(k+1)^2 - k^2 \geq 3$, and the fact that $1-J \leq 0$. We further have

$$(w_{0,k} + p - w_{j,k})^y = \frac{-1}{j^3+J} < \frac{-1}{2J} < \frac{1-J}{2J(J+3)} \leq E(t)^y = (q - w_{j,k})^y,$$

and therefore $(w_{0,k}+p)^y < q^y$. Thus, $w_{0,k} + p \in \xi_j + (0, \mathbb{R}^-, 0)$, so ξ_j is in front of Γ and as such is on the front boundary of K. □

4.2 The Translates

We now pick a number $m \in \mathbb{N}$ and construct a family $\mathcal{F}$ of $3m$ translates of K such that their union will have a cubic number of holes. This construction is identical to that in Sect. 3:

First, for $k \in [m]$, we define a translate C_k such that the edge $u_1 - E_k$ of C_k coincides with the edge E_k of K, that is

$$C_k = K + c_k, \quad \text{where} \quad c_k = w_{0,k} - (u_1 - w_{0,k+1}).$$

Again we denote the vertical facet formed by $u_1 - E_k$ and $u_0 - E_k$ of C_k as R_k.

Next, for $j \in [m]$, we define a translate B_j of K as follows:

$$B_j = K + b_j, \quad \text{where} \quad b_j = -w_{j,0}.$$

In other words, the path ξ_j of B_j lies in the (x, y)-plane, the vertex $v_{j,k}$ of B_j lies on E_k.

For the third group of translates we need to determine a sufficiently small $\varepsilon > 0$. First, observe that the points $w_{j,k}$, for $j, k \in [m]$, do not lie on K. Let $\varepsilon_1 > 0$ be smaller than the distance of $w_{j,k}$ to K, for all $j, k \in [m]$. Second, consider the m^2

points $v_{j,k} + b_j$ for $j, k \in [m]$ on the path γ. Let ε_2 be the shortest distance between any two of these points.

Consider now the segment $w_{j,k}w_{j,k+1}$, for some $j, k \in [m]$. It touches K in the point $v_{j,k}$; the rest of the segment lies entirely outside K. This implies that there is an $\varepsilon_{j,k} > 0$ such that any line parallel to $w_{j,k}w_{j,k+1}$ at distance less than $\varepsilon_{j,k}$ intersects K only within a neighborhood of $v_{j,k}$ of radius $\varepsilon_2/3$.

We choose $\varepsilon < \varepsilon_1$ and $\varepsilon < \varepsilon_{j,k}$, for all $j, k \in [m]$. With this choice of $\varepsilon > 0$, we can finally define, for $i \in [m]$:

$$A_i = K + a_i, \quad \text{where } a_i = \left(0, 0, -\tfrac{i}{m}\varepsilon\right).$$

Our family $\mathcal{F}$ is

$$\mathcal{F} = \{A_1, \ldots, A_m,\ B_1, \ldots, B_m,\ C_1, \ldots, C_m\}.$$

4.3 The Nerve

To every family $\mathcal{X} = \{X_1, X_2, \ldots, X_n\}$ of n sets is associated a collection of subfamilies $\mathcal{N}(\mathcal{X})$, called the *nerve* of $\mathcal{X}$, defined as follows:

$$\mathcal{N}(\mathcal{X}) = \Big\{\mathcal{Y} \subseteq \mathcal{X} : \bigcap_{X \in \mathcal{Y}} X \neq \emptyset\Big\}.$$

In a sense, the nerve is a natural generalization of the *intersection graph*. In Sect. 5, we will count the number of holes in the union of $\mathcal{F}$ by computing the rank of certain matrices defined in terms of its nerve. We now give an explicit description of the nerve of the family $\mathcal{F}$. Consider the following subfamilies of $\mathcal{F}$:

$$\begin{aligned}
\Delta_1 &= \{A_1, \ldots, A_m, B_1, \ldots, B_m\}, \\
\Delta_2 &= \{B_1, \ldots, B_m, C_1, \ldots, C_m\}, \\
\Delta_{i,k} &= \{A_i, C_k, C_{k+1}\} && \text{for } (i, k) \in [m] \times [m-1], \\
\Delta_{i,j,k} &= \{A_i, B_j, C_k\} && \text{for } (i, j, k) \in [m]^3.
\end{aligned}$$

Lemma 2 *The set of inclusion-maximal subfamilies in $\mathcal{N}(\mathcal{F})$ is*

$$\mathcal{M} = \{\Delta_1, \Delta_2\} \cup \{\Delta_{i,k} : (i, k) \in [m] \times [m-1]\} \cup \{\Delta_{i,j,k} : (i, j, k) \in [m]^3\}.$$

Proof To check that the subfamilies in $\mathcal{M}$ are in $\mathcal{N}(\mathcal{F})$, we find a point in the intersection of each of them. Specifically, these points are

$$\begin{aligned}
w_{0,k+1} + a_i &\in A_i \cap C_k \cap C_{k+1}, \\
v_{j,k} + a_i + b_j &\in A_i \cap B_j \cap C_k, \\
(0, -1, 0) &\in \bigcap\{A_1, \ldots, A_m, B_1, \ldots, B_m\}, \\
(\zeta_2, 2, -1) &\in \bigcap\{B_1, \ldots, B_m, C_1, \ldots, C_m\}.
\end{aligned}$$

Verifying that these points are in the subfamilies as indicated is a tedious calculation, which we leave to Sect. 6.

Now, let σ be a maximal subfamily in $\mathcal{N}(\mathcal{F})$. If σ does not contain any C_k, then $\sigma \subseteq \Delta_1$ and by maximality $\sigma = \Delta_1$. Similarly, if σ does not contain any A_i, then $\sigma \subseteq \Delta_2$ and by maximality $\sigma = \Delta_2$.

We can therefore assume that $A_i, C_k \in \sigma$. By definition of K and $\mathcal{F}$ we have $A_i \cap C_k = E_k + a_i$. Since $A_i \cap C_k$ and $A_{i'} \cap C_k$ are parallel segments for $i' \neq i$, σ cannot contain $A_{i'}$.

Assume now that σ contains no B_j. The segments E_k and $E_{k'}$ intersect if and only if k and k' differ by one. It follows that σ is either $\Delta_{i,k}$ or $\Delta_{i,k-1}$.

In the final case, σ contains some B_j, for $j \in [m]$. The segment $E_k + a_i - b_j$ is parallel to $w_{j,k}w_{j,k+1}$ at distance at most $\varepsilon < \varepsilon_{j,k}$, and so it intersects K only in a neighborhood of $v_{j,k}$ of radius at most $\varepsilon_2/3$. It follows that $E_k + a_i$ intersects $B_j = K + b_j$ only in a neighborhood of the same radius around the point $v_{j,k} + b_j$. But, the shortest distance between these points is ε_2, and so these neighborhoods are disjoint. It follows that σ contains no other $B_{j'}$, for $j' \neq j$.

Since the point $w_{0,k} + a_i$ lies at distance at most ε from $w_{0,k}$, but the point $w_{j,k} = w_{0,k} - b_j$ has distance larger than $\varepsilon_1 > \varepsilon$ from K, we have $A_i \cap C_{k-1} \cap C_k = w_{0,k} + a_i \notin B_j$, and so $C_{k-1} \notin \sigma$. For the same reason $C_{k+1} \notin \sigma$. It follows that $\sigma = \{A_i, C_k, B_j\} = \Delta_{i,j,k}$. □

5 Counting Holes in the Union via the Nerve

In this section we use homology to count the number of holes in a union of convex objects and prove Theorem 1; we first illustrate our arguments by giving a new proof of Kovalev's upper bound [14]. We start by recalling some standard topological machinery.

5.1 Homology and Betti Numbers

An *abstract simplicial complex* Δ with vertex set V is a set of subsets of V closed under taking subsets: if $\sigma \in \Delta$ and $\tau \subseteq \sigma$ then $\tau \in \Delta$. An element $\sigma \in \Delta$ is called a *simplex*; the *dimension* of a simplex is its cardinality minus 1, so singletons are simplices of dimension 0, pairs are simplices of dimension 1, etc. A simplex of dimension i is called an *i-simplex* for short. The *vertices* of a simplex $\sigma \in \Delta$ are the singletons contained in σ. Note that the nerve of a family of convex sets is an abstract simplicial complex.

Let Δ be an abstract simplicial complex on a *totally ordered* vertex set V. The ith real chain space of Δ, denoted $\mathcal{C}_i(\Delta)$, is the real vector space spanned[2] by the i-simplices of Δ. For $i \in \mathbb{N}$, the *ith boundary map* $\partial_i : \mathcal{C}_i(\Delta) \to \mathcal{C}_{i-1}(\Delta)$ is the linear map defined on a basis of $\mathcal{C}_i(\Delta)$ as follows. For any i-simplex $\sigma = \{v_0, v_1, \cdots, v_i\} \in \Delta$

[2] In other words, the i-dimensional simplices of Δ form a basis of the vector space $\mathcal{C}_i(\Delta)$, which consists of formal sums of i-simplices, each simplex being assigned a real coefficient.

with $v_0 < v_1 < \ldots < v_i$,

$$\partial_i(\sigma) = \sum_{j=0}^{i} (-1)^j (\sigma \setminus \{v_j\}).$$

That is, ∂_i maps every i-dimensional simplex to an element of $\mathcal{C}_{i-1}(\Delta)$, namely an alternating sum of its facets. Observe that $\partial_i \circ \partial_{i+1} = 0$, so that $\operatorname{im} \partial_{i+1} \subseteq \ker \partial_i$. The ith *simplicial homology group* $H_i(\Delta, \mathbb{R})$ of Δ is defined as the quotient $\ker \partial_i / \operatorname{im} \partial_{i+1}$ and the ith *Betti number* $\beta_i(\Delta)$ of Δ is the dimension of $H_i(\Delta, \mathbb{R})$, hence:

$$\beta_i(S) = \dim \ker \partial_i - \operatorname{rank} \partial_{i+1}. \tag{1}$$

If X is a topological space, one can define, in a similar but more technical way, the *singular homology groups* of X and its *Betti numbers*. We do not recall those definitions (the interested reader is referred to [10,16]) but emphasize two facts that will be useful:

(i) $\beta_0(X)$ is the number of connected components of X, assuming X admits a cell decomposition.
(ii) If $\mathcal{X} = \{X_1, X_2, \ldots, X_n\}$ is a family of convex objects in $\mathbb{R}^d$ and $U = \bigcup_{i=1}^n X_i$ then $H_i(U, \mathbb{R}) \simeq H_i(\mathcal{N}(\mathcal{X}), \mathbb{R})$; as a consequence, U and $\mathcal{N}(\mathcal{X})$ have the same Betti numbers. This follows from the classical *Nerve Theorem* of Borsuk [6].

5.2 Counting Holes

We can now relate the number of holes in the union of a family of convex objects to one particular Betti number of its nerve.

Lemma 3 *If $\mathcal{X} = \{X_1, X_2, \ldots, X_n\}$ is a family of compact convex objects in $\mathbb{R}^d$, with $d \geq 2$, then the number of holes of $U = \bigcup_{i=1}^n X_i$ is $\beta_{d-1}(\mathcal{N}(\mathcal{X})) + 1$.*

Proof The number of holes of U is, by definition, the number of connected components of $\mathbb{R}^d \setminus U$, which is $\beta_0(\mathbb{R}^d \setminus U)$. As $d \geq 2$, for any compact locally contractible subset $T \subseteq \mathbb{S}^d$, Alexander duality gives

$$\beta_{d-1}(T) = \beta_0(\mathbb{S}^d \setminus T) - 1.$$

Identifying the d-sphere with the one-point compactification of d-space, $\mathbb{S}^d \simeq \mathbb{R}^d \cup \{\infty\}$, we have that

$$\beta_0(\mathbb{R}^d \setminus U) = \beta_0(\mathbb{S}^d \setminus U) = \beta_{d-1}(U) + 1,$$

and by the Nerve Theorem, $\beta_{d-1}(U) = \beta_{d-1}(\mathcal{N}(\mathcal{X}))$. □

As an illustration let us see how a version of the upper bound of Kovalev [14] for compact convex objects immediately follows from Lemma 3:

Corollary 3 *The number of holes in the union of n compact convex objects in* $\mathbb{R}^d$ *is at most* $\binom{n}{d} + 1$.

Proof Let $\mathcal{X}$ be a family of n compact convex objects in $\mathbb{R}^d$. By Lemma 3, the union of the members of $\mathcal{X}$ has $\beta_{d-1}(\mathcal{N}(\mathcal{X})) + 1$ holes. Let ∂_i denote the ith boundary operator of $\mathcal{N}(\mathcal{X})$. By definition, β_{d-1} is the dimension of the quotient of the vector space $\ker \partial_{d-1}$ by the vector space $\operatorname{im} \partial_d$. Now, $\ker \partial_{d-1}$ is contained in the space $\mathcal{C}_{d-1}(\mathcal{N}(\mathcal{X}))$ spanned by the $(d-1)$-simplices of $\mathcal{N}(\mathcal{X})$; since $\mathcal{N}(\mathcal{X})$ has n vertices, it has at most $\binom{n}{d}$ simplices of dimension $d-1$ and $\ker \partial_{d-1}$ therefore has dimension at most $\binom{n}{d}$. This dimension can only go down by taking the quotient by $\operatorname{im} \partial_d$, so $\beta_{d-1}(\mathcal{N}(\mathcal{X})) \leq \binom{n}{d}$. □

5.3 The Number of Holes in the Union of $\mathcal{F}$

We now prove Theorem 1 by using Lemma 3.

Proof (Theorem 1) Kovalev [14] already established that any union of n convex objects in $\mathbb{R}^3$ has $O(n^3)$ holes. Hence this bound applies to families of translates. It remains to prove that this bound is tight by constructing a family whose union has $\Omega(n^3)$ holes. Let K denote the convex body, and let $\mathcal{F}$ denote the family of $n = 3m$ translates of K constructed above. Let $U = \bigcup_{X \in \mathcal{F}} X$.

Recall that the maximal simplices of $\mathcal{N}(\mathcal{F})$ are identified by Lemma 2. By Lemma 3, the number of holes of U is $\beta_2(\mathcal{N}(\mathcal{F})) + 1$ which equals, by Eq. (1), $\dim \ker \partial_2 - \operatorname{rank} \partial_3 + 1$, where ∂_i denotes the ith boundary map of $\mathcal{N}(\mathcal{F})$. We compute $\beta_2(\mathcal{N}(\mathcal{F}))$ by computing explicitly a basis for $\ker \partial_2$ and a basis for $\operatorname{im} \partial_3$.

To compute a basis of $\operatorname{im} \partial_3$, let $\mathcal{S}$ denote the set of 3-simplices of $\mathcal{N}(\mathcal{F})$ containing B_1 and let $\mathcal{T}$ stand for the set of images of the simplices of $\mathcal{S}$ under ∂_3:

$$\mathcal{S} = \{\sigma : |\sigma| = 4 \text{ and } B_1 \in \sigma\} \quad \text{and} \quad \mathcal{T} = \{\partial_3 \sigma : \sigma \in \mathcal{S}\}.$$

Observe that $\mathcal{T}$ is a linearly independent family. Indeed, for any $\sigma \in \mathcal{S}$, the 2-simplex $\sigma \setminus \{B_1\}$ has non-zero coefficient in $\partial_3(\sigma)$ but has zero coefficient in $\partial_3(\tau)$ for every $\tau \in \mathcal{S} \setminus \{\sigma\}$. To see that $\mathcal{T}$ spans $\operatorname{im} \partial_3$, let σ be a 3-simplex of $\mathcal{N}(\mathcal{F})$. If $B_1 \in \sigma$ then $\sigma \in \mathcal{S}$ and so $\partial_3(\sigma) \in \mathcal{T}$. If $B_1 \notin \sigma$, since $\partial_3 \circ \partial_4 = 0$ we have

$$\partial_3 \circ \partial_4(\sigma \cup \{B_1\}) = \lambda \partial_3(\sigma) + \sum_{X \in \sigma} \lambda_v \partial_3(\sigma \cup \{B_1\} \setminus \{X\}) = 0$$

where λ and the λ_v are in $\{\pm 1\}$. This implies that $\partial_3(\sigma)$ is a sum of $\pm \partial_3(\tau)$ with $\tau \in \mathcal{S}$, and thus lies in the span of $\mathcal{T}$. Therefore, as claimed, $\mathcal{T}$ is a basis of $\operatorname{im} \partial_3$.

Now, let $\mathcal{S}'$ denote the set of 2-simplices in $\mathcal{F}$ that contain B_1 and let $\mathcal{T}' = \{\partial_2 \sigma : \sigma \in \mathcal{S}'\}$. The same arguments yield that $\mathcal{T}'$ is a basis of $\operatorname{im} \partial_2$. Also let $\mathcal{S}''$ denote the set of all 2-simplices contained in $\mathcal{F}$.

We can finally compute $\beta_2(\mathcal{N}(\mathcal{F}))$ using the rank-nullity theorem,

$$\begin{aligned}\beta_2(\mathcal{N}(\mathcal{F})) &= \text{nullity } \partial_2 - \text{rank } \partial_3\\ &= \dim \mathcal{C}_2(\mathcal{N}(\mathcal{F})) - \text{rank } \partial_2 - \text{rank } \partial_3\\ &= |\mathcal{S}''| - |\mathcal{T}'| - |\mathcal{T}|.\end{aligned}$$

Counting all quadruples in Δ_1 or Δ_2 that contain B_1 (taking care that some quadruples appear both in Δ_1 and Δ_2) we have

$$|\mathcal{T}| = |\mathcal{S}| = 2\binom{2m-1}{3} - \binom{m-1}{3}.$$

Next, counting all triples in $\mathcal{F}$ that contain B_1 we have

$$|\mathcal{T}'| = |\mathcal{S}'| = \binom{3m-1}{2}.$$

Then, counting all triples contained in $\mathcal{N}(\mathcal{F})$, which are the triples in $\Delta_{i,j,k}$ plus $\Delta_{j,k}$ plus the triples in Δ_1 or Δ_2 (accounting for triples belonging to both Δ_1 and Δ_2), we have

$$|\mathcal{S}''| = m^3 + m(m-1) + 2\binom{2m}{3} - \binom{m}{3}.$$

Finally, using $\binom{a}{b} - \binom{a-1}{b} = \binom{a-1}{b-1}$, we have

$$\begin{aligned}\beta_2(\mathcal{N}(\mathcal{F})) &= |\mathcal{S}''| - |\mathcal{T}| - |\mathcal{T}'|\\ &= m^3 + m(m-1) + 2\binom{2m}{3} - \binom{m}{3} - 2\binom{2m-1}{3} + \binom{m-1}{3} - \binom{3m-1}{2}\\ &= m^3 + m(m-1) + 2\binom{2m-1}{2} - \binom{m-1}{2} - \binom{3m-1}{2}\\ &= m^3 - m.\end{aligned}$$

So $\mathcal{F}$ is a family of $3m$ translates of a convex body in $\mathbb{R}^3$, and the union has $m^3 - m + 1$ holes. □

6 Witness Points for Lemma 2

In this section we argue that the four sets of translates of K considered in Lemma 2 have non-empty intersection. We show this by exhibiting an explicit point in the intersection of each set, in the following four claims. We will need the bounds

$$1 < \zeta_3 < 5/4, \quad 3/2 < \zeta_2 < 7/4.$$

Claim For $i \in [m]$ and $k \in [m-1]$ we have $w_{0,k+1} + a_i \in A_i \cap C_k \cap C_{k+1}$.

Proof Since $w_{0,k+1} \in K$, $w_{0,k+1} + a_i \in A_i$. Furthermore, $w_{0,k+1}$ coincides with the point $u_1 - w_{0,k}$ of C_k and the point $u_1 - w_{0,k+1}$ of C_{k+1}. Since the vertical segment below $w_{0,k+1}$ lies in $C_k \cap C_{k+1}$ and $\varepsilon < \zeta_3$, the claim holds. □

Claim For $(i, j, k) \in [m]^3$ we have $v_{j,k} + a_i + b_j \in A_i \cap B_j \cap C_k$.

Proof The point $v_{j,k} + b_j$ lies in B_j and on the edge E_k, the top edge of the facet R_k of C_k. The translation a_i is a small vertical translation, so $v_{j,k} + b_j + a_i$ lies inside R_k and therefore in C_k, and lies in the interior of B_j. Finally, since $v_{j,k} + b_j \in E_k \subset K$, the point $v_{j,k} + b_j + a_i \in A_i$. □

Claim We have $(0, -1, 0) \in \bigcap\{A_1, \dots, A_m, B_1, \dots, B_m\}$.

Proof The point $p = (0, -1, 0)$ lies on the bottom edge of K connecting $w_{0,0} = (0, 0, 0)$ and $u_0 - w_{0,\infty} = (0, -2 - \zeta_3, 0)$. Since the vertical segment of length one with bottom end p lies in K and $\varepsilon < 1$, we have $p - a_i \in K$ and therefore $p \in A_i$, for $i \in [m]$.

The length of the top edge of K connecting $w_{\infty,0} = (0, -\zeta_2, \zeta_3)$ and $u_1 - w_{0,\infty} = (0, -2 - \zeta_3, \zeta_3)$ is $2 + \zeta_3 - \zeta_2 > 1$, and so the path $p + \eta$ lies in K. This implies that $p - b_j = p + w_{j,0} \in K$, and therefore $p \in B_j$, for $j \in [m]$. □

Claim We have $p = (\zeta_2, 2, -1) \in \bigcap\{B_1, \dots, B_m, C_1, \dots, C_m\}$.

Proof For $j \in [m]$, we have $-b_j = w_{j,0} \in \{0\} \times [-\zeta_2, -1] \times [1, \zeta_3]$. This gives

$$p - b_j \in \{\zeta_2\} \times [2 - \zeta_2, 1] \times [0, \zeta_3 - 1] \subset \{\zeta_2\} \times [-2, 1] \times [0, \zeta_3 - 1].$$

The four corners of this rectangle are

$$u_0 = (\zeta_2, -2, 0) \in K, \qquad q_1 = (\zeta_2, -2, \zeta_3 - 1) \in [u_0, u_1],$$
$$q_2 = (\zeta_2, 1, 0) \in [u_0, w_{0,\infty}], \qquad q_3 = (\zeta_2, 1, \zeta_3 - 1).$$

Since $w_{0,\infty} = (\zeta_2, \zeta_3, 0)$, $w_{1,\infty} = (\zeta_2, \zeta_3 - 1, 1)$, and $0 < \zeta_3 - 1 < 1$, we have

$$q_3 = (2 - \zeta_3) w_{0,\infty} + (\zeta_3 - 1) w_{1,\infty} \in K,$$

and so the entire rectangle lies in K. This implies $p - b_j \in K$, and so $p \in K + b_j = B_j$.

For $k \in [m]$, we have

$$\begin{aligned} p - c_k &= p + u_1 - w_{0,k} - w_{0,k+1} = (2\zeta_2, 0, \zeta_3 - 1) - w_{0,k} - w_{0,k+1} \\ &= \Bigl(2\zeta_2 - 2\sum_{t=1}^{k} t^{-2} - (k+1)^{-2}, -2\sum_{t=1}^{k} t^{-3} - (k+1)^{-3}, \zeta_3 - 1\Bigr) \\ &\in [0, 2\zeta_2 - 2 - 1/4] \times [-2\zeta_3, -2 - 1/8] \times \{\zeta_3 - 1\} \\ &\subset [0, \zeta_2 - 1/2] \times [-5/2, -1] \times \{\zeta_3 - 1\}. \end{aligned}$$

Here we used $\zeta_2 < 7/4$ and $\zeta_3 < 5/4$. The four corners of this rectangle are

$$q_1 = (0, -5/2, \zeta_3 - 1), \qquad q_2 = (0, -1, \zeta_3 - 1),$$
$$q_3 = (\zeta_2 - 1/2, -5/2, \zeta_3 - 1), \qquad q_4 = (\zeta_2 - 1/2, -1, \zeta_3 - 1).$$

We will argue that all four corners lie in K, implying that $p - c_k \in K$, and so $p \in K + c_k = C_k$.

For q_1 and q_2 this is obvious, as they lie inside the convex hull of the points

$$w_{0,0} = (0, 0, 0), \qquad w_{1,0} = (0, -1, 1),$$
$$u_0 + w_{0,\infty} = (0, -2 - \zeta_3, 0), \qquad u_1 + w_{0,\infty} = (0, -2 - \zeta_3, \zeta_3).$$

The point q_4 lies on the segment q_2q_5, where $q_5 = (\zeta_2, -1, \zeta_3 - 1)$, and q_5 lies in the convex hull of $w_{0,\infty}$, $w_{1,\infty}$, u_0, and u_1.

Finally, the point q_3 lies on the facet $u_0u_1 - E_0$ of K. Indeed, the corners of this facet are

$$u_0 = (\zeta_2, -2, 0), \qquad u_0 - w_{0,1} = (\zeta_2 - 1, -3, 0),$$
$$u_1 = (\zeta_2, -2, \zeta_3), \qquad u_1 - w_{0,1} = (\zeta_2 - 1, -3, \zeta_3).$$

□

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