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Antiprismlessness, or: Reducing Combinatorial Equivalence to Projective Equivalence in Realizability Problems for Polytopes

Michael Gene Dobbins[1]

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Abstract This article exhibits a 4-dimensional combinatorial polytope that has no antiprism, answering a question posed by Bernt Lindström. As a consequence, any realization of this combinatorial polytope has a face that it cannot rest upon without toppling over. To this end, we provide a general method for solving a broad class of realizability problems. Specifically, we show that for any first-order semialgebraic property that faces inherit, the given property holds for some realization of every combinatorial polytope if and only if the property holds from some projective copy of every polytope. The proof uses the following result by Below. Given any polytope with vertices having algebraic coordinates, there is a combinatorial "stamp" polytope with a specified face that is projectively equivalent to the given polytope in all realizations.

Keywords Polytopes · Realizability · Prismatoids · Lattices · Real closed fields

Mathematics Subject Classification 52B11 · 52B05 · 52B40 · 03C10 · 14P10

1 Introduction

The combinatorial type of a polytope is defined by the partial ordering of its face lattice. We generally "see" a partial ordering by drawing its Hasse diagram; see Fig. 1. If we draw the Hasse diagram of the face lattice of a polytope, we may observe that

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Michael Gene Dobbins
mdobbins@binghamton.edu

[1] Department of Mathematical Sciences, Binghamton University, Binghamton, NY 13902-6000, USA

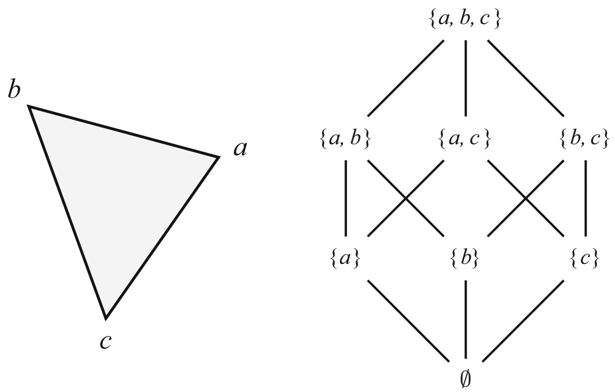

Fig. 1 A triangle and a Hasse diagram of its face lattice

it resembles the 1-skeleton of a larger polytope. For example, the Hasse diagram of a simplex's face lattice is the 1-skeleton of a hypercube standing on a vertex. The 1-skeleton alone does not uniquely determine the combinatorial type of a polytope, but there is a natural extension of the Hasse diagram that does, the intervals of a poset ordered by inclusion. When the original poset is a combinatorial polytope, the resulting poset of intervals shares some basic properties with combinatorial polytopes, such as being a lattice and satisfying Euler's formula [11].

In 1971 Lindström asked whether the intervals of a polytope's face lattice always form a new combinatorial polytope [12]. In this article, we will see that this is not the case. Moreover, we will construct a 4-polytope such that the poset of intervals of its face lattice is not the combinatorial type of any polytope. An equivalent question appears in the 1967 edition of Grünbaum's text book [8, Sect. 4.8, Prob. 19], and has applications in linear optimization [5]. Namely, does every polytope have an antiprism? An antiprism is the combinatorial dual of the interval polytope (see Fig. 2). Anders Björner showed that every 3-polytope has an antiprism [4]; see also [9].

Broadie gave sufficient conditions for a polytope to have an antiprism [5]. These conditions ask for a perfectly centered realization of the original polytope [6]. Perfectly centered has a nice physical interpretation; it says a polytope can rest on any face without toppling over, assuming the polytope filled with some inhomogeneous material so that its center of mass is at a prespecified point. Since every 3-polytope has an antiprism, every combinatorial 3-polytope has a realization that can rest on each of its faces. Here, we will give both necessary and sufficient conditions for the realizability of a polytope's antiprism, then construct a 4-polytope without an antiprism. Consequently, every realization of this 4-polytope has some face on which it cannot rest. Note, however, that the face on which it cannot rest is not necessarily a facet.

As part of this construction, we provide a technique for answering questions of the following form. Does a certain geometric property hold for some realization of every combinatorial polytope? Such questions are made difficult by the universality theorem for polytopes. The universality theorem states that for any primary basic semialgebraic set X, there exists a poset with realization space (modulo isometries)

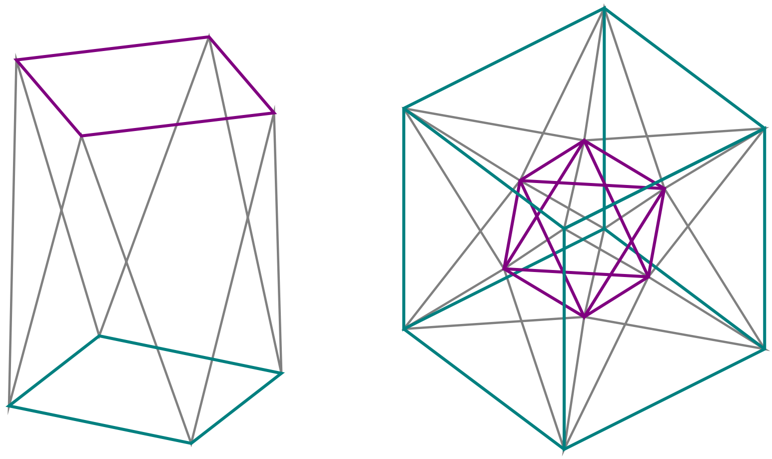

Fig. 2 *Left*: The antiprism of a square. *Right*: The antiprism of a cube

that is homotopy equivalent to X. Jürgen Richter-Gebert showed that universality holds even for polytopes of dimension 4 [13]. As a consequence, searching for a realization of a certain combinatorial polytope that satisfies a certain geometric property, can be as hard as searching for a point in a semialgebraic set. This may be difficult, since a semialgebraic set may be disconnected or have holes or other unwanted features for a search space.

For a broad class of properties of polytopes, we show that the problem of determining whether such a property holds for some realization of every combinatorial type can be reduced to determining whether the property holds for some realization of every projective type, which is a finer partition (Fig. 3). This is a considerable improvement since, in contrast to the realization spaces of polytopes with fixed combinatorial type, the space of polytopes (up to isometry) with fixed projective type is convex. We will also see that, when such a realization does not always exist, there is a gap of at most 2 between the lowest dimension where this fails for combinatorial types and the lowest dimension where it fails for projective types.

For a geometric property that is general enough to be relevant in any dimension, if the property holds for a polytope, then in many cases, it holds for the polytope's faces as well. We say faces projectively inherit a property when, for any face of a polytope with this property, some projective copy of the face has the property. Theorem 2.2 gives a reduction from combinatorial type to projective type for any property faces projectively inherit provided we restrict ourselves to polytopes with vertices having algebraic coordinates. Using basic model theory, Theorem 2.3 gives the same reduction from combinatorial type to projective type for polytopes with real coordinates provided the property considered is first-order semialgebraic. Either the property must be first-order semialgebraic, or the polytopes must be algebraic.

An example of a property that faces inherit is, "The polytope's vertices have rational coordinates". Trivially, if this is true of a polytope then it is also true of its faces. This reduces the question "Can every combinatorial 4-polytope be realized with ratio-

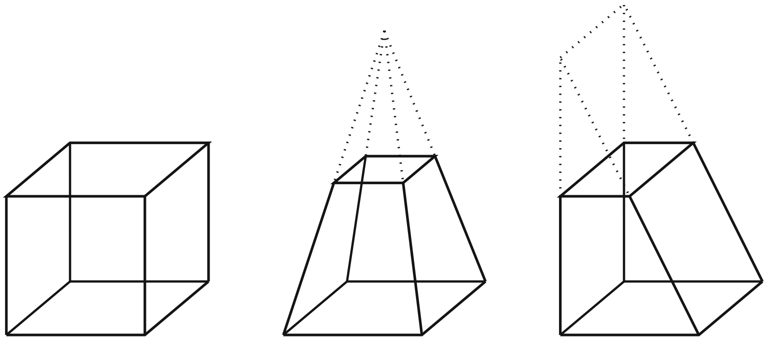

Fig. 3 Three combinatorially equivalent polytopes. Only the left two are projectively equivalent

nal coordinates?" to the question "Does every polygon with algebraic vertices have a projective copy with rational vertices?". Answering the first question was a considerable hurdle that paved the way for Gale duality and the universality theorem [17]. The answer to the second question, however, is easily seen to be no. Just consider the regular pentagon (Fig. 4).

This reduction is a consequence of a construction we call a stamp of a polytope. Adiprasito and Padrol have used stamp polytopes in a similar way to construct a polytope that is not combinatorially equivalent to any subpolytope of a stacked polytope [1]. In $\mathbb{R}^3$ it is known that faces of polyhedra are prescribable [2]. That is, given a realization of a face of a combinatorial 3-polytope, it is always possible to extend this to a realization of the entire polytope. This does not hold in higher dimensions [10, 16], and a stamp gives the strongest possible violation of this for polytopes in general. A stamp is a combinatorial polytope that forces a specified face to have a fixed projective type in all realizations. In his doctoral thesis, Below constructed a stamp for any projective type of polytope having vertices with coordinates in the real algebraic completion of the rationals [3]. Below's thesis was unfortunately never published, but the author has provided an alternate stamp construction in a longer version of this article available online, arXiv:1307.0071.

1.1 Organization of the Paper

Section 2 deals with the reduction from combinatorial equivalence to projective equivalence. Section 3 proves some basic results about prismoids, then shows that the existence of a balanced pair is equivalent to the existence of an antiprism, and then presents a polytope without an antiprism. Finally, Sect. 4 leaves the reader with some open questions.

1.2 Terminology and Notation

In this article, we assume that polytope's faces are indexed by a poset with the order of indices in the poset indicating containment of faces. We denote the face of a polytope

P that has index f by $F = \text{face}(P, f)$, and we may refer to F as the face f of P. We say polytopes are combinatorially equivalent (or have the same combinatorial type) when their faces are indexed by the same poset, and we will make use of the implied correspondence between their faces. Note that a polytope may have non-trivial symmetry, and some properties considered depend on the indexing of the faces in a way that is not preserved by these symmetries.

An invertible projective transformation on $\mathbb{R}^n$ is called a projectivity, and we say polytopes P and Q are projectively equivalent (or have the same projective type) when they are combinatorially equivalent and there exists a projectivity π such that for each index f, $\pi(\text{face}(P, f)) = \text{face}(Q, f)$. Note that the restriction of φ to P is bounded, but does not necessarily preserve orientation. We may also say Q is a projective copy of P.

We denote column vectors by $[x_1; x_2; \ldots; x_n]$, and we may include a set among the entries of a vector to indicate a Cartesian product. For example, $[P; 1] = \{[x_1; \ldots; x_n; 1] : [x_1; \ldots; x_n] \in P\}$. A brief glossary of notation follows.

$\mathbb{R}_{\text{alg}}$	The real algebraic closure of the rationals.
$f \wedge g$	The lattice meet; e.g. logical conjunction "and".
$f \vee g$	The lattice join; e.g. logical disjunction "or".
$\bot$ ($\top$)	The least (greatest) element of a bounded poset.
$\mathcal{P}^*$	The poset $\mathcal{P}$ with order reversed.
$\mathcal{P} \times \mathcal{Q}$	The product of posets, $(a, x) \leq (b, y)$ when $a \leq b$ and $x \leq y$.
$\mathcal{P}_1 \wedge \mathcal{P}_2$	The common refinement of a pair of sublattices.
$\text{fl}(P)$	The poset indexing the face lattice of a polytope P.
$\text{face}(P, f)$	The face of a polytope P labeled by f.
P°	The relative interior of a set P.
P^*	The polar of a centered polytope (or cone) P.
$\text{conv}(X)$	The convex hull of X.
$P_1 \Cup P_2$	The convex join, $\text{conv}(P_1 \cup P_2)$.
$\text{cone}(P, f)$	The cone emanating from the origin over the face f of P.
$\text{ncone}(P, f)$	The normal cone of the face f of P.
$\text{nfan}(P)$	The normal fan of P.
$P \overset{\text{comb}}{\sim} Q$	P and Q are combinatorially equivalent.
$[P]_{\text{comb}}$	All polytopes that are combinatorially equivalent to P.
$P \overset{\text{proj}}{\sim} Q$	P and Q are projectively equivalent.
$[P]_{\text{proj}}$	All polytopes that are projectively equivalent to P.

2 Combinatorial and Projective Equivalence

In this section we prove the reduction for realizability problems from combinatorial equivalence to projective equivalence using stamps. The *stamp* of an algebraic polytope P is the pair $(\mathcal{S}_P, f_P)$ implied by the following theorem of Below [3, p. 134], see also arXiv:1307.0071 for an alternate proof.

Theorem 2.1 *Given an algebraic d-polytope P, there exists a combinatorial $(d+2)$-polytope $\mathcal{S}_P$ with a specified face $f_P \in \mathcal{S}_P$ such that for any realization S of $\mathcal{S}_P$, the specified face is projectively equivalent to the given polytope,* $\text{face}(S, f_P) \overset{\text{proj}}{\sim} P$.

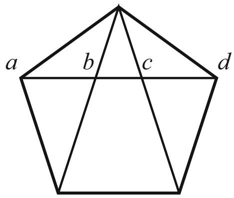

Fig. 4 For any projective copy of a regular pentagon, the cross-ratio $(a, b \mid c, d) = \frac{1+\sqrt{5}}{2}$ is irrational, so the vertices cannot have rational coordinates

The stamp helps us answer questions about properties that faces inherit, or more generally the following class of predicates. Let ψ be a predicate of several algebraic polytopes of the same combinatorial type. We say the face $f \in \mathcal{P}$ *projectively inherits* ψ when, if $P_1, \ldots, P_n$ are realizations of $\mathcal{P}$ such that $\psi(P_1, \ldots, P_n)$ is true, then there are projectivities $\pi_1, \ldots, \pi_n$ such that $\psi(\pi_1(\text{face}(P_1, f)), \ldots, \pi_n(\text{face}(P_n, f)))$ is true. Recall that a ridge of a polytope is a face of co-dimension 2.

Theorem 2.2 *Let ψ be a predicate of several algebraic polytopes of the same combinatorial type that ridges projectively inherit. Then, ψ holds for some realization of every combinatorial type of polytope if and only if it holds for some realization of every algebraic projective type. Moreover, there can be a gap of at most* 2 *between the lowest dimension of a combinatorial type where ψ always fails and the lowest dimension of an algebraic projective type where ψ always fails,*

$$\begin{aligned}
&\forall v_1, \ldots, v_m \in \mathbb{R}^{d+2}_{\text{alg}}\ \exists P_1, \ldots, P_n \in [\text{conv}(v_1, \ldots, v_m)]_{\text{comb}}.\ \psi(P_1, \ldots, P_n)\\
&\quad \Rightarrow \forall v_1, \ldots, v_m \in \mathbb{R}^{d}_{\text{alg}}\ \exists P_1, \ldots, P_n \in [\text{conv}(v_1, \ldots, v_m)]_{\text{proj}}.\ \psi(P_1, \ldots, P_n)\\
&\quad \Rightarrow \forall v_1, \ldots, v_m \in \mathbb{R}^{d}_{\text{alg}}\ \exists P_1, \ldots, P_n \in [\text{conv}(v_1, \ldots, v_m)]_{\text{comb}}.\ \psi(P_1, \ldots, P_n).
\end{aligned}$$

Proof Since projective equivalence is finer than combinatorial equivalence, we have the 'if' direction trivially. For the other direction, let ψ be a predicate that ridges projectively inherit and suppose ψ holds for some realizations of every combinatorial polytope. Consider an algebraic polytope P. Since the stamp $\mathcal{S}_P$ has realizations where ψ holds, it must also hold for some projective copies of the face f_P of each of these realizations, and these faces are all projectively equivalent to P by Theorem 2.1. Thus, ψ holds for some projective copies of P. Also, since $\mathcal{S}_P$ is 2 dimensions higher than P, if ψ holds for some realizations of every combinatorial polytope up to dimension $d + 2$, then it holds for some projective copies of every algebraic polytope up to dimension d. □

Generally it is common to consider polytopes in $\mathbb{R}^d$, so it would be nicer if Theorem 2.2 were not restricted to polytopes in $\mathbb{R}^d_{\text{alg}}$. If we simply removed the restriction to algebraic polytopes, the resulting claim would be false. For example, there are projective types of polygons that never have algebraic coordinates, but every combinatorial polytope can be realized with algebraic coordinates.

Instead, we replace the restriction to algebraic polytopes with a restriction on the property considered. Specifically, we require the predicate ψ to be a first-order semialgebraic. We say ψ is a *first-order semialgebraic property* when it is expressible in the language of real closed fields as a formula depending on combinatorial type. That is, ψ is first-order semialgebraic when, for any combinatorial d-polytope $\mathcal{P}$ with m vertices, there is a formula using first-order logic ($\forall, \exists, \neg, \wedge, \vee$) and basic algebraic operations ($+, \cdot$) and relations ($=, \leq$) on dmK variables such that $\psi(P_1, \ldots, P_K)$ is true for polytopes $P_1, \ldots, P_K$ if and only if the coordinates of the polytopes' vertices satisfy the formula. ψ cannot involve quantification over sets of numbers or trigonometric functions for example. For an accessible review of real closed fields and model theory see [7, Sect. IV.23].

Theorem 2.3 *If ψ is a first-order semialgebraic property of several polytopes of the same combinatorial type such that ridges always projectively inherit ψ, then ψ holds for some realizations of every combinatorial type of polytope if and only if it holds for some realizations of every projective type. Moreover, there can be a gap of at most* 2 *dimensions.*

Proof Briefly, Theorem 2.3 follows from the fact that $\mathbb{R}$ and $\mathbb{R}_{\text{alg}}$ are elementarily equivalent [14], and both realizability and projective equivalence are expressible in the language of real closed fields.

We first express realizability and projective equivalence in the language of real closed fields. Note that we will exclusively use $\wedge$ for logical conjunction in this proof. For a formula ϕ with free variable x, and a formula θ, recall $\phi[\theta/x]$ denotes the formula where x is replaced by θ. We will also use conventional notation in $\mathbb{R}^d$ as abbreviations for the formulas that can easily be written in the language of real closed fields. For example, $x - y = z$ should be understood as $x = z + y$.

First we write a formula $\psi'_{\mathcal{P}}$ that says there are realizations of $\mathcal{P}$ where $\psi_{\mathcal{P}}$ holds. For this, write a formula $\rho_{\mathcal{P}}$ with dmK free variables $v_{i,j,k}$ for each coordinate of each vertex $\mathbf{v}_{j,k}$ of each polytope P_k that says these are indeed vertex coordinates of a polytope of type $\mathcal{P}$. Let $\mathcal{F}_1, \ldots, \mathcal{F}_n$ be the set vertices in each facet of $\mathcal{P}$.

$$\rho_{\mathcal{P}} := \exists \mathbf{a}_{1,1} \ldots \exists \mathbf{a}_{n,K} . \nu_{\mathcal{P}}.$$

$$\nu_{\mathcal{P}} := \bigwedge_{k=1}^{K} \Big(\bigwedge_{v_j \in \mathcal{F}_i} \langle \mathbf{a}_{i,k}, \mathbf{v}_{j,k} - \boldsymbol{\tau}_k \rangle = 1 \Big) \wedge \Big(\bigwedge_{v_j \notin \mathcal{F}_i} \langle \mathbf{a}_{i,k}, \mathbf{v}_{j,k} - \boldsymbol{\tau}_k \rangle < 1 \Big).$$

$$\boldsymbol{\tau}_k := \frac{1}{m} \sum_{j=1}^{m} \mathbf{v}_{j,k}.$$

Note that $\boldsymbol{\tau}_k$ is a formula for a translation vector that centers P_k. The formula $\nu_{\mathcal{P}}$ includes free variables for the half-spaces supporting each facet of a centered translation of the polytope, and says that the vertices of a facet are on the boundary of its supporting half-space and the rest of the vertices are in the interior of this half-space. Now write a formula $\psi'_{\mathcal{P}}$ asserting the existence of a realization where $\psi_{\mathcal{P}}$ holds.

$$\psi'_{\mathcal{P}} := \exists \mathbf{v}_{1,1} \ldots \exists \mathbf{v}_{m,K} . \, \rho_{\mathcal{P}} \wedge \psi_{\mathcal{P}} \, .$$

Next, write a formula $\chi_{\mathcal{P}}$ saying for K polytopes with combinatorial type $\mathcal{P}$ that there is a projective copy where predicate $\psi_{\mathcal{P}}$ holds. For this we represent projectivities π_k on $\mathbb{R}^d$ by $(d+1) \times (d+1)$ matrices M_k acting on homogeneous coordinates.

$$\chi_{\mathcal{P}} := \exists M_1 \dots \exists M_K \exists x_{1,1} \dots \exists x_{m,K} . \mu_{\mathcal{P}}, \quad M_k := \begin{bmatrix} A_k & b_k \\ c_k^* & 1 \end{bmatrix},$$

$$\mu_{\mathcal{P}} := \bigwedge_{j,k=1,1}^{m_{\mathcal{P}},K} \big(x_{j,k} \cdot (\langle \mathbf{c}_k, \mathbf{w}_{j,k} \rangle + 1) = 1\big) \wedge (\rho_{\mathcal{P}} \wedge \psi_{\mathcal{P}})[x_{j,k} \cdot (A_k \mathbf{w}_{j,k} + \mathbf{b}_k)/\mathbf{v}_{j,k}].$$

Note that the formula $\mu_{\mathcal{P}}$ is defined by replacing each coordinate of $\mathbf{v}_{j,k}$ in the formula $\rho_{\mathcal{P}} \wedge \psi_{\mathcal{P}}$ with a formula for the corresponding coordinate of $\pi_k(\mathbf{v}_{j,k}) := x_{j,k}\mathbf{u}_{1:d}$ where $\mathbf{u} = M_k[v_{j,k}; 1]$ and $x_{j,k} = u_{d+1}^{-1}$. Now write a formula $\chi'_{\mathcal{P}}$ asserting that every realization has a projective copy where $\psi_{\mathcal{P}}$ holds.

$$\chi'_{\mathcal{P}} := \forall \mathbf{v}_{1,1} \dots \forall \mathbf{v}_{m_{\mathcal{P}},K} .\ \rho_{\mathcal{P}} \Rightarrow \chi_{\mathcal{P}} .$$

In both $\mathbb{R}$ and $\mathbb{R}_{\text{alg}}$ we have immediately that the existence of a realization of every projective type where the predicate holds implies the existence of such a realization for every combinatorial type. For the other direction, suppose there is some combinatorial d-polytope $\mathcal{P}$ with realizations in $\mathbb{R}^d$ such that ψ does not hold for any projective copies in $\mathbb{R}^d$. That is, $\mathbb{R} \models \neg\chi'_{\mathcal{P}}$. Since $\mathbb{R}$ and $\mathbb{R}_{\text{alg}}$ are elementarily equivalent, we have $\mathbb{R}_{\text{alg}} \models \neg\chi'_{\mathcal{P}}$, which asserts the existence of an algebraic polytope P where ψ does not hold for any algebraic projective copies. Let $\mathcal{S}_P$ be the combinatorial $(d+2)$-polytope that is the stamp of P. Then, $\mathbb{R}_{\text{alg}} \models \neg\psi'_{\mathcal{S}_P}$, and therefore $\mathbb{R} \models \neg\psi'_{\mathcal{S}_P}$. Thus, we have found a combinatorial $(d+2)$-polytope such that ψ does not hold for any realization in $\mathbb{R}^{d+2}$. □

3 Antiprisms

A polytope is a *prismoid* when every vertex of the polytope is in one of two nonintersecting faces, which we call the *bases* of the prismoid. That is, every prismoid P is of the form

$$P = B_0 \uplus B_1 = \{t_0 B_0 + t_1 B_1 : t_i \geq 0,\ t_0 + t_1 = 1\},$$

where B_0, B_1 are disjoint faces. The *sides* of the prismoid are faces that are not contained in either base, along with the trivial side $\perp$. When a combinatorial polytope is a prismoid, we call it a combinatorial prismoid. Some examples of prismoids are pyramids, tents, prisms, and antiprisms.

We define a purely combinatorial construction, called an abstract prismoid. The definition is motivated by the fact that a face of a prismoid is determined by its intersection with each of the bases. For bounded posets $\mathcal{B}_0, \mathcal{B}_1$, an *abstract prismoid* $\mathcal{P}$ with these bases is a bounded subposet of the categorical product $\mathcal{B}_0 \times \mathcal{B}_1$ such that the bases themselves are included as $(f_0, \perp), (\perp, f_1) \in \mathcal{P}$ for all $f_i \in \mathcal{B}_i$. All faces that are not in a base and the face $\perp = (\perp, \perp)$ are sides of $\mathcal{P}$, denoted

$$\text{side}(\mathcal{P}) := \{(f_0, f_1) \in \mathcal{P} : (f_0{=}\perp) \Leftrightarrow (f_1{=}\perp)\}.$$

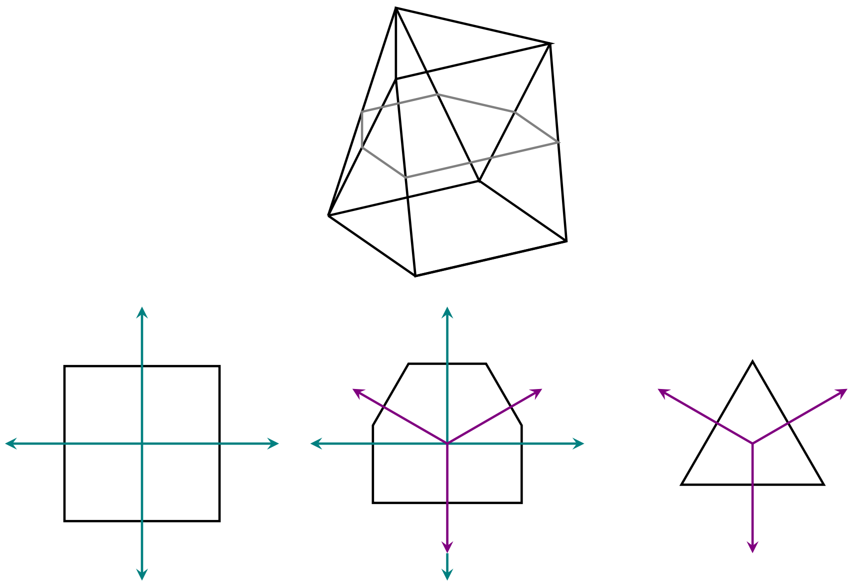

Fig. 5 *Top*: A prismoid with a *triangular* and *square* base. *Bottom*: The prismoid bases and a *horizontal* slice with the common refinement of the normal fans of the bases

Lemma 3.1 *Every combinatorial prismoid is isomorphic to an abstract prismoid. And, an abstract prismoid $\mathcal{P} \subset \mathcal{B}_0 \times \mathcal{B}_1$ can be realized if and only if there are realizations B_i of the bases $\mathcal{B}_i$ such that the combinatorial type of the common refinement of the normal fans of the B_i is the dual of the sides of $\mathcal{P}$,*

$$\mathrm{fl}(\mathrm{nfan}(B_0) \wedge \mathrm{nfan}(B_1))^* = \mathrm{side}(\mathcal{P}).$$

Moreover, the set of all realizations of $\mathcal{P}$ is the set of all polytopes that are projectively equivalent to $[B_0; 0] \cup [B_1; 1]$ for some B_i satisfying the above (*Fig.* 5).

Proof For the first part, every combinatorial prismoid is a bounded poset, and every face can be uniquely identified by its intersection with the bases, and the bases are faces, so every combinatorial prismoid is isomorphic to an abstract prismoid. The second part follows from the fact that we can project a prismoid so that its bases are in parallel hyperplanes, in which case a horizontal slice between these hyperplanes is a weighted Minkowski sum of the bases, and the normal fan of the Minkowski sum of a pair of polytopes is the common refinement of their normal fans [16, Prop. 7.12].

Specifically, for the 'if' direction of the second part, suppose we have such realizations B_0, B_1. Recall that every non-empty face of a polytope (in particular $P = [B_0; 0] \cup [B_1; 1] \subset \mathbb{R}^d$) is the solution set of some linear optimization problem. Specifically, for a face f these are the optimization problems with linear objective function in the relative interior of the normal cone of f. Notice that the optimal solutions in $[B_i; i]$ depend only on the restriction of the linear objective function $v^* = [w^*\ c]$ to the first $d-1$ coordinates. For $q_i \in [B_i; i]$, we have $v^* q_0 = w^* q_0$ and $v^* q_1 = w^* q_1 + c$.

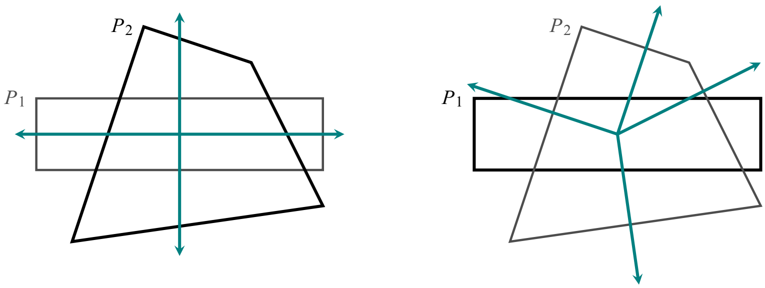

Fig. 6 *Left* A pair of quadrilaterals that are balanced, balance(P_1, P_2). *Right* The same pair of quadrilaterals in reverse order is not balanced, $\neg$ balance(P_2, P_1)

For optimal solutions q_i in B_i to objective w^*, setting $c = w^*(q_0 - q_1)$ we get $v^* q_0 = v^* q_1$. Hence, a non-trivial pair of faces (f_0, f_1) defines a face of P if and only if there is a vector in the relative interior of the normal cone of the faces f_0 of B_0 and f_1 of B_1. Therefore, side(fl(P)) is the common refinement of the normal fans of B_0 and B_1.

For the 'only if' direction suppose we have a realization P of some abstract prismoid $\mathcal{P} \subset \mathcal{B}_0 \times \mathcal{B}_1$. Let $B_0 = \text{face}(P, (\top, \bot))$ and $B_1 = \text{face}(P, (\bot, \top))$. Then there is some projective transformation π sending B_i into the hyperplane $\{x : x_d = i\}$. Hence $\pi(P) = [B_0'; 0] \uplus [B_1'; 1]$ for some realizations B_i' of $\mathcal{B}_i$, and again side$(\mathcal{P})$ is the common refinement of the normal fans of B_0' and B_1'. ◻

Recall from the introduction that an (antiprism) interval polytope of P is defined as a polytope with face lattice consisting of the intervals of fl(P) ordered by (reverse) inclusion. The *abstract antiprism* of a bounded poset $\mathcal{P}$ is the poset $\{(g, f^*) \in \mathcal{P} \times \mathcal{P}^* : g \leq f\}$. When an abstract antiprism can be realized we call it a combinatorial antiprism.

We say a polytope is *centered* when it contains the origin in its interior. We call an ordered pair of polytopes (P_1, P_2) *balanced*, when they are centered, have the same combinatorial type, and the relative open normal cone of a face g of P_1 intersects the relative open face f of P_2 if and only if f is greater than g,

$$\text{balance}(P_1, P_2) := \forall g, f \neq \bot.\ (\text{ncone}(P_1, g)^\circ \cap \text{face}(P_2, f)^\circ \neq \emptyset \Leftrightarrow g \leq f)\,.$$

Observe that balance(P_1, P_2) implies balance(P_2^*, P_1^*), but that balance is not a symmetric relation, see Fig. 6.

Theorem 3.2 *A combinatorial polytope has an antiprism if and only if it has a balanced pair. Moreover, if* balance(P_0, P_1) *then* $[P_0; 0] \uplus [P_1^*; 1]$ *is an antiprism of* P_0.

Proof By Lemma 3.1, $\mathcal{P}$ has an antiprism if and only if there are realizations P_0 of $\mathcal{P}$ and P_1^* of $\mathcal{P}^*$ such that the common refinement of their normal fans are indexed by minimal pairs $(g, f^*) \in \mathcal{P} \times \mathcal{P}^*$ such that $g \leq f$. And, in this case $[P_0; 0] \uplus [P_1^*; 1]$ is an antiprism. The common refinement of the normal fans consists of minimal non-empty intersections, so this is equivalent to the statement, "$\mathrm{ncone}(P_0, g)^\circ$ and $\mathrm{ncone}(P_1^*, f^*)^\circ$ intersect if and only if $g \leq f$". Since $\mathrm{ncone}(P_1^*, f^*)^\circ = \mathrm{cone}(P_1, f)^\circ = \mathbb{R}_{\geq 0}\,\mathrm{face}(P_1, f)^\circ$ and P_1 realizes $\mathcal{P}$, this is equivalent to $\mathrm{balance}(P_0, P_1)$. □

We say a polytope is *perfectly centered* when the orthogonal projection of the origin into the affine closure of each face is in the relative interior of that face. In particular, if a perfectly centered polytope has full dimension, then it is centered. Note this has the following physical interpretation. A polytope is perfectly centered when it can rest on any face without toppling over, assuming its center of mass is at the origin. This is also equivalent to the polytope being balanced with itself.

Lemma 3.3 *A polytope is perfectly centered if and only if it is balanced with itself.*

Proof By [5, Thm. 2.1], a polytope P is perfectly centered if and only if $[P; 0] \uplus [P^*; 1]$ is an antiprism of P, and by Theorem 3.2, this holds if and only if $\mathrm{balance}(P, P)$. □

Unlike 4 dimensions, where the problem of realizing polytopes has no easy solution, the situation is much simpler in 3 dimensions [13]. Every 3-polytope has a particularly nice realization called a midscribed polytope [15]. Recall that a 3-polytope is midscribed when every edge is tangent to the unit sphere.

Theorem 3.4 *Midscribed polytopes are perfectly centered. Hence, every combinatorial 3-polytope has an antiprism.*

Proof Each edge of a midscribed polytope is tangent to the unit sphere, so the orthogonal projection of the origin into the line spanning that edge is exactly the point of tangency. Hence, the perfectly centered condition holds for edges. For each facet of a midscribed polytope, the plane spanning that facet intersects the unit ball in a disk, and each edge of the facet is tangent to the disk. In other words, each facet circumscribes the disk where it intersects the unit ball. The orthogonal projection of the origin into the plane spanning the facet is the center of this disk. Hence, the condition holds for facets. Since midscribed polytopes correspond to disk packings of the sphere in this way, we may assume the polytope is centered, otherwise apply an appropriate conformal map so that the disks are not all contained in any single hemisphere [15]. For vertices, the condition is trivial. Thus, midscribed polytopes are perfectly centered, and since every combinatorial 3-polytope has such a realization, by Lemma 3.3 every 3-polytope has a realization that is balanced with itself, so by Theorem 3.2 every combinatorial 3-polytope has an antiprism. □

We now set about constructing a 4-dimensional polytope without an antiprism. By Lemma 3.1 this is equivalent to finding a combinatorial 4-polytope without a balanced pair. We use Theorem 2.3 to reduce this problem to finding an polygon that cannot be balanced by projective transformations, Lemma 3.6. To use Theorem 2.3, we need to show that faces projectively inherit balance, Lemma 3.8, and that balance is an first-order semialgebraic property, which may be observed directly from the definition of balance.

Since balance is defined in terms of polar duality, it will be helpful to recall how the polar dual of a polytope is transforms when applying a projective transformation in the primal. We represent a projective transformation π acting on a vector $x \in \mathbb{R}^d$ by a matrix $M = [A, b; c^*, 1]$ acting on homogeneous coordinates by

$$\pi(x) = \begin{bmatrix} A & b \\ c^* & 1 \end{bmatrix}_{\mathbb{P}} (x) = \frac{Ax + b}{c^*x + 1}.$$

For π as above, we denote

$$\pi^* = \left(\begin{bmatrix} -I & 0 \\ 0 & 1 \end{bmatrix} M^* \begin{bmatrix} -I & 0 \\ 0 & 1 \end{bmatrix} \right)_{\mathbb{P}} = \begin{bmatrix} A^* & -c \\ -b^* & 1 \end{bmatrix}_{\mathbb{P}},$$

and we call $\pi^{-*} := (\pi^*)^{-1} = (\pi^{-1})^*$ the polar transformation of π.

Proposition 3.5 *For a centered polytope P and a projectivity π such that $\pi(P)$ is centered, bounded, and has the same orientation as P,*

$$\pi(P)^* = \pi^{-*}(P^*).$$

Proof Since π is bounded and preserves orientation on P, we have $\forall p \in P.\, c^*p+1 > 0$. Since $\forall x \in \pi(P)^*, \forall p \in P^\circ.\ \langle x, \pi p\rangle < 1$, and in particular $\langle x, \pi \mathbf{0}\rangle = x^*b < 1$, we have $-b^*x + 1 > 0$. Thus,

$$\begin{aligned} \pi(P)^* &= \big\{x : \forall q \in \pi(P), \langle x, q\rangle \le 1\big\} \\ &= \big\{x : \forall p \in P, \langle x, \pi p\rangle \le 1\big\} \\ &= \big\{x : \forall p \in P, \langle x, \tfrac{Ap+b}{c^*p+1}\rangle \le 1\big\} \\ &= \big\{x : \forall p \in P, \langle x, Ap\rangle \le \langle c, p\rangle + 1 - \langle x, b\rangle\big\} \\ &= \big\{x : \forall p \in P, \langle \tfrac{A^*x-c}{-b^*x+1}, p\rangle \le 1\big\} \\ &= \big\{x : \forall p \in P, g\langle \pi^*x, p\rangle \le 1\big\} \\ &= \big\{\pi^{-*}(y) : \forall p \in P, \langle y, p\rangle \le 1\big\} \\ &= \pi^{-*}(P^*). \end{aligned}$$ □

Suppose we are given a pair of polygons, and we want to find a balanced pair of projective copies. We start by reducing the space of transformations we need to consider. Since applying a projective transformation to one polygon and applying the polar transformation to the other polygon preserves balance, we can reduce the problem to applying a projective transformation to only one polygon and keeping the other polygon fixed. Furthermore, a projective transformation consists of an affine part and a perspectivity, but a perspectivity applied to a single vector only scales that vector. To see this, compare the matrix representation of a projective transformation to that of the affine part of the same transformation

$$\begin{bmatrix} A & b \\ c^* & 1 \end{bmatrix}_{\mathbb{P}} (x) = \frac{Ax + b}{c^*x + 1}, \qquad \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix}_{\mathbb{P}} (x) = \frac{Ax + b}{1}.$$

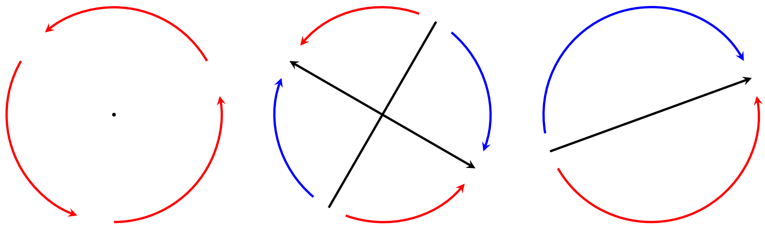

Fig. 7 From the *left*, the way vectors turn by rotating, stretching, and translating

Since scaling vectors by positive values does not change the cone of positive linear combinations of those vectors, and balance depends on the intersection of cones, we can reduce the problem further to balancing a pair of polygons by applying an affine transformation to one of the polygons.

We will now see informally why an affine transformation cannot always balance polygons. Two polygons P_0, P_1 are balanced if and only if the direction vectors of the vertices of P_0 and P_1^* are interleaved around the unit circle. That is, the vectors alternate around the unit circle between belonging to one polygon and the polar of the other. For an affine transformation to balance one polygon with another, it may have to change some of these direction vectors to make them alternate. If we think of moving a transformation continuously from the identity to one that balances the pair, along the way the vectors will turn clockwise or counter-clockwise on the circle. The main idea is to construct a pair of polygons that require such an affine transformation to turn too many direction vectors alternately clockwise and counter-clockwise.

To get an idea of how many direction vectors is too many we decompose an affine transformation into parts and see how many vectors each part can handle. Assume the transformation preserves orientation. Consider the special orthogonal linear part (rotating), symmetric positive definite linear part (stretching), and translational part of an orientation preserving affine transformation. The orthogonal part turns all vectors in the same way. The spd part divides the circle into four quadrants where direction vectors turn alternately clockwise and counter-clockwise. And, the translational part divides the circle into two halves where direction vectors turn the opposite way. Naively adding this up we get seven regions; see Fig. 7. We construct a polygon such that an affine transformation must to turn eight vectors alternately clockwise and counter-clockwise to balance the polygon with a copy of itself, which we then show is impossible.

With these limitations in mind, let G be the polygon with vertices $(8, 5)$, $(7, 7)$ and all permutations and changes of sign: $(8, 5)$, $(7, 7)$, $(5, 8)$, $(-5, 8)$, $(-7, 7)$, $(-8, 5)$, $(-8, -5)$, $(-7, -7)$, $(-5, -8)$, $(5, -8)$, $(7, -7)$, $(8, -5)$ (Fig. 8).

Lemma 3.6 *No two projective copies of G are balanced.*

Proof The polar G^* has vertices $(\frac{1}{8}, 0)$, $(\frac{2}{21}, \frac{1}{21})$ and all permutations and changes of sign. Balance is invariant under positive scaling, so we scale G^* by 21 for convenience, giving vertices $(\frac{21}{8}, 0)$, $(2, 1)$ instead. We see that G is not perfectly centered since that would require the slope m of the outward normal vector of the edge between

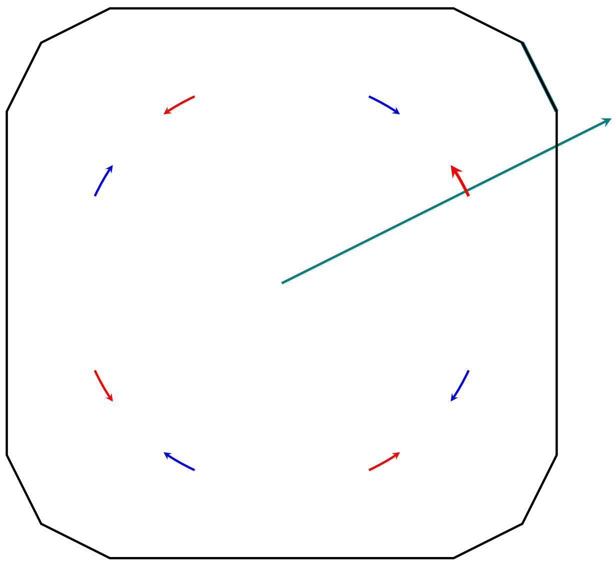

Fig. 8 A polygon that cannot be balanced with itself by affine transformations. The *thick red arrow* indicates how the normal ray of the thick edge must turn to intersect that edge. Similarly, the seven other *red* and *blue arrows* indicate how the normal rays of other edges must turn to balance an affine copy of the polygon with itself

(8, 5) and (7, 7) to be between the slopes of the vertices, $\frac{5}{8} < m < 1$, but the slope is $m = \frac{1}{2} < \frac{5}{8}$ as seen from the vertex (2, 1) of $21G^*$. By construction, the reflection group of G and G^* is the same as that of the unit square, and is given by the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

These transformations give us a total of eight places where the perfectly centered condition is violated. Consider an affine transformation T acting on $21G^*$ by

$$T\begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} s \\ t \end{bmatrix}.$$

For balance$(G, T^{-1}G)$ = balance$(T(21G^*)^*, G)$ to hold, we must have $a, d > 0$. For $a > 0$, this is because $T\left(\frac{21}{8}, 0\right)$ must point to the right and $T\left(-\frac{21}{8}, 0\right)$ must point to the left,

$$T\left(\frac{21}{8}, 0\right)_1 = \frac{21}{8}a + s > 0, \quad T\left(-\frac{21}{8}, 0\right)_1 = -\frac{21}{8}a + s < 0.$$

If $s \leq 0$ then the first inequality implies $a > 0$, and if $s \geq 0$ then the second inequality implies $a > 0$. The same holds for d because of the corresponding inequalities in the 2nd coordinate.

Additionally, the image of (2, 1) must have slope greater than $\frac{5}{8}$, and this must also be the case for T conjugated by all elements of the reflection group. The image of (2, 1) by all conjugates is

$$\begin{bmatrix} a2+b1+s \\ c2+d1+t \end{bmatrix}, \begin{bmatrix} d2+c1+t \\ b2+a1+s \end{bmatrix}, \begin{bmatrix} d2-c1-t \\ -b2+a1+s \end{bmatrix}, \begin{bmatrix} a2-b1-s \\ -c2+d1+t \end{bmatrix},$$

$$\begin{bmatrix} a2+b1-s \\ c2+d1-t \end{bmatrix}, \begin{bmatrix} d2+c1-t \\ b2+a1-s \end{bmatrix}, \begin{bmatrix} d2-c1+t \\ -b2+a1-s \end{bmatrix}, \begin{bmatrix} a2-b1+s \\ -c2+d1-t \end{bmatrix}.$$

For the first of these vectors the slope requirement is given by the following inequality

$$\frac{T(2,1)_2}{T(2,1)_1} = \frac{c2+d1+t}{a2+b1+s} > \frac{5}{8}.$$

Equivalently, $-10a - 5b + 16c + 8d - 5s + 8t > 0$. Putting the inequalities we get from all these slope requirements with the sign requirements of a, d together we get the matrix inequality

$$\begin{bmatrix} -10 & -5 & 16 & 8 & -5 & 8 \\ 8 & 16 & -5 & -10 & 8 & -5 \\ 8 & -16 & 5 & -10 & 8 & 5 \\ -10 & 5 & -16 & 8 & 5 & 8 \\ -10 & -5 & 16 & 8 & 5 & -8 \\ 8 & 16 & -5 & -10 & -8 & 5 \\ 8 & -16 & 5 & -10 & -8 & -5 \\ -10 & 5 & -16 & 8 & -5 & -8 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ s \\ t \end{bmatrix} > 0\,.$$

Finding a solution to this inequality amounts to finding a vector in the column space of the matrix that has all positive entries. The columns of this matrix, however, are all orthogonal to [1; 1; 1; 1; 1; 1; 1; 1; 8; 8], which has all positive entries, so the column span of the matrix is outside of the open positive orthant, which implies that no values for a, b, c, d, s, t satisfy all of these inequalities. Therefore, there is no affine transformation T such that balance$(G, T^{-1}G)$.

If there were projectivities π_1, π_2 such that balance$(\pi_1(G), \pi_2(G))$, then we would have balance$(G, \pi_1^*\pi_2(G))$, and the affine part of $\pi_1^*\pi_2$ would balance G with itself, which we have just seen to be impossible. Thus, no two projective copies of G are balanced. □

We will now show that faces projectively inherit balance. For some intuition why, notice that an interval polytope of $\mathcal{P}$ has among its faces all of the interval polytopes of the faces of $\mathcal{P}$. By Theorem 3.2, if a combinatorial polytope $\mathcal{P}$ has a balanced pair, then it has an interval polytope, which means all of its faces also have interval polytopes. By the other direction of Theorem 3.2, we get a balanced pair for any face of $\mathcal{P}$. Hence, if a combinatorial polytope has a balanced pair, then each of its faces also has a balanced pair. But to show projective inheritance, we need slightly more. We need the faces of a balanced pair (P_1, P_2) to have projective copies that are balanced.

To find projective copies of some face of a balanced pair (P_1, P_2), we perform the natural geometric analog of the above argument. We construct the antiprism A with bases P_1, P_2, then the polar dual of a facet of A^* will be an antiprism A_f having balanced projective copies of the corresponding faces f of P_1, P_2 as bases.

Before proving Lemma 3.8, we present the basic algebraic rules relating face cones, normal cones, and polar duality for polytopes when working in subspace and direct sums of vector spaces. For a cone C in a vector space V, let $C^\diamond \subset \operatorname{span}(C)^*$ denote the polar dual of C regarded as a polytope in the vector subspace spanned by C. Clearly if C has full dimension then $C^\diamond = C^*$. Given two vector spaces V_1, V_2, let $V_1 \oplus V_2$ denote the direct product. For $a_i \in V_i^*$ let $a_1 \oplus a_2$ be the linear functional on $V_1 \oplus V_2$ defined by

$$[a_1 \oplus a_2](x_1, x_2) = a_1(x_1) + a_2(x_2).$$

Note that if $V_1, V_2 \subset V$ are complimentary vector subspaces, then $V_1 \oplus V_2 = V$, and we may consider V_1^* as being orthogonal to V_2 in this sense that $V_1^* \oplus \mathbf{0} \subset V^*$ is the subspace of linear functionals that vanish on V_2. Recall that the set of translations of an affine space X form a vector space, which we denote $\operatorname{trans}(X)$. For a polytope $P \subset X$, the tangent cone $\operatorname{ncone}(P, f)^* \subset \operatorname{trans}(X)$ is the cone generated by all translations that send some point on the face $\operatorname{face}(P, f)$ to a point in P, and the normal cone $\operatorname{ncone}(P, f) \subset \operatorname{trans}(X)^*$ is the polar dual of the tangent cone. For a flat $V \supset P$ of X and a point $p \in V$, let $i_{V,p}(P)$ denote P regarded as a polytope in the vector space V with origin p.

Proposition 3.7 *For cones C_1, C_2 spanning complementary subspaces,*

$$\begin{aligned}
\operatorname{fl}(C_1 + C_2) &= \operatorname{fl}(C_1) \times \operatorname{fl}(C_2),\\
(C_1 + C_2)^* &= C_1^\diamond \oplus C_2^\diamond,\\
\operatorname{face}(C_1 + C_2, (f_1, f_2)) &= \operatorname{face}(C_1, f_1) + \operatorname{face}(C_2, f_2).
\end{aligned}$$

For a centered polytope P with face indices $g \leq f$ and $r > 0$,

$$\begin{aligned}
\operatorname{fl}(\operatorname{cone}(P, f)) &= [\bot, f],\\
\operatorname{face}(\operatorname{cone}(P, f), g) &= \operatorname{cone}(\operatorname{face}(P, g)),\\
\operatorname{ncone}(P, f) &= \operatorname{cone}(P^*, f^*),\\
\operatorname{ncone}(\operatorname{face}(P, f), g) &= \operatorname{face}(\operatorname{ncone}(P, g)^*, f)^*,\\
\operatorname{cone}([P; r])^* &= \operatorname{cone}([P^*; -1/r]).
\end{aligned}$$

For a polytope P in affine space with a face f, a point $p \in P$, and a subspace $V \supset P$,

$$\text{ncone}(i_{V,p}(P), f)^* = p + \text{ncone}(P, f)^*.$$

We do not include a proof of Proposition 3.7 here. These rules may easily but tediously be verified by the reader.

Lemma 3.8 *Faces projectively inherit the predicate* balance.

Proof Let P_1, P_2 be centered realizations of $\mathcal{P}$ such that balance(P_1, P_2). By Lemma 3.1, P_1, P_2 are base faces of an antiprism $A = [P_1; 1] \uplus [P_2^*; -1]$. Note that the faces of A are

$$\text{face}(A, (g, f^*)) = \text{face}\,([P_1; 1], g) \uplus \text{face}([P_2^*; -1], f^*)$$

for $g \leq f$, and cone$([P_1; 1], f)$ and cone$([P_2^*; -1], f^*)$ are in complementary linear subspaces.

For any $f \in \mathcal{P}$, let $A_f^* = \text{face}(A^*, (\top, f))$, and let $A_f = i_{V,p}(A_f^*)^*$ where V is the affine span of A_f^*, and p is some point in the relative interior of A_f^*. The faces of A_f are indexed by $(h, g^*) \in [\bot, f] \times [f^*, \bot^*] \subset \mathcal{P} \times \mathcal{P}^*$ such that $h \leq g$. Let $F_i = \text{face}(P_i, f)$, $\tilde{F}_1 = \text{face}(A_f, (f, f^*))$, $\tilde{F}_2^* = \text{face}(A_f, (\bot, \top))$, and $\tilde{F}_2 = i_{W,q}(\tilde{F}_2^*)^*$ where W is the affine span of $\tilde{F}_2^*$ and q is some point in the relative interior of $\tilde{F}_2^*$. Observe that A_f is an antiprism of the combinatorial polytope $[\bot, f]$, so by Lemma 3.1 again, we have balance$(\tilde{F}_1, \tilde{F}_2)$.

We claim that $\tilde{F}_1, \tilde{F}_2$ are projective copies of F_1, F_2 respectively.

$$\begin{aligned}
\text{cone}(\tilde{F}_1)^* &= \text{cone}(A_f, (f, f^*))^* \\
&= \text{cone}(i_{V,p}(\text{face}(A^*, (\top, f)))^*, (f, f^*))^* \\
&= \text{ncone}(i_{V,p}(\text{face}(A^*, (\top, f))), (f^*, f))^* \\
&= p + \text{ncone}(\text{face}(A^*, (\top, f)), (f^*, f))^* \\
&= p + \text{face}(\text{ncone}(A^*, (f^*, f))^*, (\top, f)) \\
&= p + \text{face}(\text{cone}(A, (f, f^*))^*, (\top, f)) \\
&= p + \text{face}((\text{cone}(A, (f, \bot)) + \text{cone}(A, (\bot, f^*)))^*, (\top, f)) \\
&= p + \text{face}(\text{cone}(A, (f, \bot))^\diamond \oplus \text{cone}(A, (\bot, f^*))^\diamond, (\top, f)) \\
&= p + \text{face}(\text{cone}(A, (f, \bot))^\diamond, \top) \oplus \text{face}(\text{cone}(A, (\bot, f^*))^\diamond, f) \\
&= p + \text{cone}([F_1; 1])^\diamond \oplus \mathbf{0}.
\end{aligned}$$

Since V is a translate of the orthogonal compliment of cone$(A, (\bot, f^*))$, we have cone$(\tilde{F}_1) = \text{proj}_V(p + \text{cone}([F_1; 1]))$ where proj_V denotes orthogonal projection into V. Thus, $\tilde{F}_1$ and F_1 are projectively equivalent.

$$\begin{aligned}\text{cone}(\tilde{F}_2^*)^* &= \text{cone}(A_f, (\bot, \top))^* \\ &= p + \text{face}((\text{cone}(A, (\bot, \bot)) + \text{cone}(A, (\bot, \top)))^*, (\top, f)) \\ &= p + \text{face}((\mathbf{0} + \text{cone}([P_2^*; -1]))^*, (\top, f)) \\ &= p + \text{face}(\text{cone}([P_2^*; -1])^*, f) \\ &= p + \text{face}(\text{cone}([P_2; 1]), f) \\ &= p + \text{cone}([F_2; 1]).\end{aligned}$$

We also have

$$\text{cone}(\tilde{F}_2^*)^* = \text{cone}\left(i_{V,p}(\tilde{F}_2) - (1 + \|q\|^{-2})q\right),$$

where q is regarded as a vector in V with origin p. Thus, $\tilde{F}_2$ and F_2 are projectively equivalent, so the claim holds. This implies that faces projectively inherit balance. □

Theorem 3.9 *In dimensions* 4 *or more there exists a combinatorial polytope that does not have a balanced pair, antiprism, or interval polytope.*

Proof By Lemma 3.8, we can use Theorem 2.3 to reduce this problem to finding a polygon that cannot be balanced by projective transformations, and by Lemma 3.6, G is exactly such a polygon. Hence, there is a combinatorial 4-polytope that does not have a balanced pair. Specifically, the stamp $\mathcal{S}_G$ from Theorem 2.1 is such a polytope. And, by Theorem 3.2 this polytope does not have an antiprism or an interval polytope. □

4 Questions

The most apparent question is, what further applications does the stamp have? Among geometric properties of polytopes that are being studied, which of these do faces inherit, and can Theorem 2.3 be applied? Such methods have recently been employed in [1].

The stamp fixes a face of co-dimension 2 up to projectivity, but what about co-dimension 1. We saw in the introduction a 3-polytope does not impose any completion condition on its facets, but such a stamp may exist in higher dimensions. Is there a d_0 such that for any polytope P of dimension $d \geq d_0$ there is a combinatorial polytope of dimension $d+1$ such that in all realizations a specified facet is projectively equivalent to P? Or, are there other properties P could to satisfy to guarantee that such a combinatorial $(d+1)$-polytope exists?

This article was initially motivated by the question, “Is the Hasse diagram of the face lattice of any polytope the 1-skeleton of some other polytope?” but we have not fully settled that question. Theorem 3.9 says that if such a polytope exists, it is not the natural candidate, the interval polytope.

A sufficient condition was already known for a polytope to have an antiprism, that some realization is perfectly centered. Here we saw a condition that is necessary and sufficient, that some pair of realizations is balanced, but it is not immediately clear that this new condition is actually weaker. Does there exist a combinatorial polytope that has a pair of balanced realizations, but does not have a perfectly centered realization?

We have seen a variety of polytopes with various completion conditions. What sort of condition can be the completion condition of a face of a polytope? Ideally, this question would be answered by giving a formal language together with a semantic interpretation that includes polytopes among its ground types that satisfies the following. Given a set of realizations R of a combinatorial polytope $\mathcal{P}$, there is another combinatorial polytope $\mathcal{Q}$ such that R is the restriction of realizations of $\mathcal{Q}$ to a certain face, $R = \{\text{face}(Q, f) : \text{fl}(Q) = \mathcal{Q}\}$ if and only if there exists a predicate ψ in this language such that R is the set where the predicate holds $R = \{P : \psi(P) = \text{True}\}$. Of course, it would have to be possible to formulate the completion conditions already given as a predicate in such a language. In particular it must be possible to say that a polytope is fixed up to projectivity. In the other direction, any predicate of this language would have to be projectively invariant.

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