M221 **Practice Midterm** **Fall 2012**

The purpose of this document is to give you a sense of the flavor of problems to expect on the midterm, as well as the overall length. Note this is a practice midterm for a 1-hour test: if your section meets for a longer time period, then your test may be longer.

No books, no notes, no calculators. Write all your work on this test–nothing else will be graded. You must show your work.

On some problems, you are asked to use a specific method to solve the problem (for instance, "Use the definition of the derivative to find..."). On all other problems, you may use any method we've covered.

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1. (16 points) a. Give a formula for an example of a function with a jump discontinuity at 2. (If you can't do this, for partial credit draw the graph of a function with this property.)

b. Give a formula for an example of a function that is continuous on $(-\infty, \infty)$ and differentiable everywhere except at 1. (If you can't do this, for partial credit draw the graph of a function with this property.)

c. Give a formula for an example of a function f such that $\lim_{x\to 3} = \infty$. (If you can't do this, for partial credit draw the graph of a function with this property.)

2. (12 points) Find $\lim_{x\to 4^-} \frac{x-4}{|4x-x^2|}$.

3. (12 points) Find $\lim_{x\to \pi/4} \frac{1-\tan(x)}{\sin(x)-\cos(x)}$.

4. (12 points) Find all points in $[0, 2\pi]$ at which the tangent line to the curve $\cos(2x) + \sin(2x)$ is horizontal.

5. (12 points) If a rock is thrown upward on Mars with a velocity of 10 meters/second, its height in meters t seconds later is given by $y = 10t - 1.86t^2$. Find the average velocity over the time interval $[1, 2]$, and find the instantaneous velocity 3 seconds after being thrown.

6. (112 points) *Using the definition of the derivative*, find the slope of the tangent to the curve $y = 1/\sqrt{x}$ at the point $(2, 1/\sqrt{2})$.

7. (12 points) Consider a lens with focal length f. If an object is placed at distance v from the lens, then its image will be at distance u from the lens, where f, u, and v are related by

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}.$$

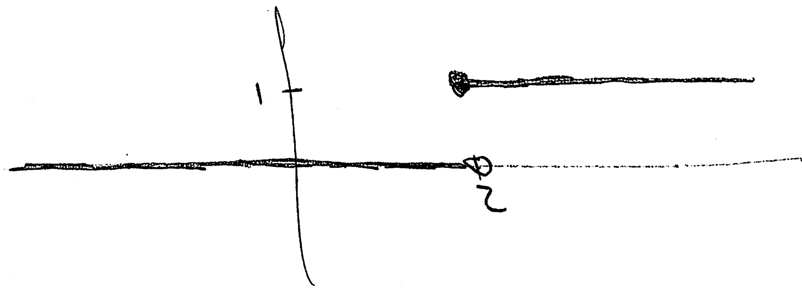
Find the rate of change of u with respect to v.

8. (12 points) A child is sitting on the ground and and flying a kite. The kite moves horizontally away from the child at 10 feet/second while staying 30 feet above the ground. Find the rate at which the angle between the kite string and the ground is decreasing when 50 feet of string has been let out.

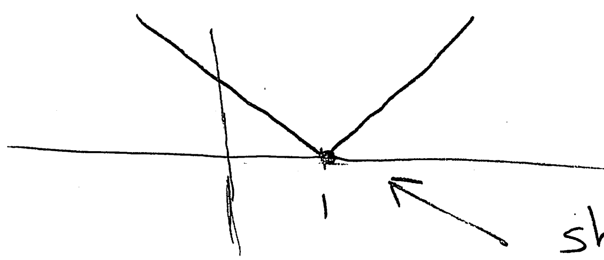
Practice Midterm Solution set

1. (a.) $f(x) = \begin{cases} 1 & \text{if } x \geq 2 \\ 0 & \text{if } x < 2 \end{cases}$.

The graph looks like this:

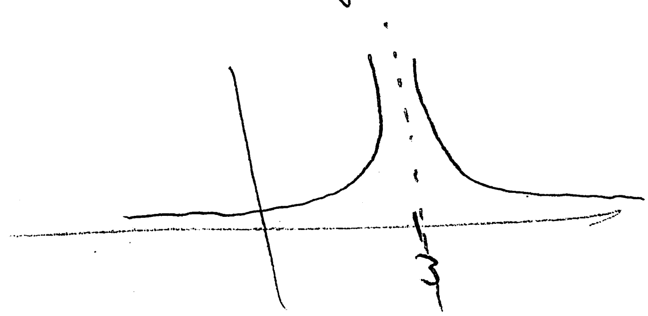


(b.) $f(x) = |x-1|$. The graph looks like this:



sharp corner at $x=1$, so $f'(1)$ doesn't exist.

(c.) $y = \dfrac{1}{(x-3)^2}$. The graph looks like this



(vertical asymptote at $x=3$).

(The even power of $(x-3)$ makes the left & right limits match at $x=3$.)

2. When x is slightly less than 4, we have $4x - x^2 > 0$, so $|4x - x^2| = 4x - x^2$.

Then

$$\lim_{x \to 4^-} \frac{x-4}{|4x - x^2|} = \lim_{x \to 4^-} \frac{x-4}{4x - x^2}$$

$$= \lim_{x \to 4^-} \frac{x-4}{(4-x)\cdot x}$$

$$= \lim_{x \to 4^-} \frac{x-4}{-(x-4)\cdot x}$$

$$= \lim_{x \to 4^-} \frac{1}{-x} = -\frac{1}{4}.$$

3. $$\lim_{x\to\frac{\pi}{4}} \frac{1-\tan x}{\sin x - \cos x}$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{1-\frac{\sin x}{\cos x}}{\sin x - \cos x}$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{\frac{\cos x - \sin x}{\cos x}}{\frac{\sin x - \cos x}{1}}$$ invert + multiply

$$= \lim_{x\to\frac{\pi}{4}} \frac{\cos x - \sin x}{(\cos x)(\sin x - \cos x)}$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{-(\sin x - \cos x)}{(\cos x)(\sin x - \cos x)}$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{-1}{\cos x} = \frac{-1}{\frac{\sqrt{2}}{2}} = \frac{-2}{\sqrt{2}} = -\sqrt{2}.$$

4. $y = \cos(2x) + \sin(2x)$, so

$y' = -2\sin(2x) + 2\cos(2x)$. Setting $y' = 0$,

we get $0 = -2\sin(2x) + 2\cos(2x)$

$\Rightarrow 2\sin(2x) = 2\cos(2x)$

$\Rightarrow \tan(2x) = 1$

$\Rightarrow 2x = \frac{\pi}{4}$ or $2\pi = \frac{5}{4}\pi$

(over)

$$\Rightarrow \quad x = \frac{\pi}{8} \quad \text{or} \quad x = \frac{5\pi}{8}.$$

5. $v(0) = 10$ m/s, so $y(t) = 10t - 1.86t^2$.

So, $v(t) = \frac{dy}{dt} = 10 - (2)(1.86)t$.

average velocity $= \frac{y_2 - y_1}{t_2 - t_1} = \frac{y(2) - y(1)}{2-1}$

$$= \frac{\left[10(2) - 1.86(2)^2\right] - \left[10(1) - 1.86(1)^2\right]}{2-1}.$$

instantaneous velocity at $t = 3$ is $v(3)$, and

$$v(3) = \left.\frac{dy}{dt}\right|_{t=3} = 10 - (2)(1.86)(3).$$

6. $\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, which, for $f(x) = \frac{1}{\sqrt{x}}$,

$$= \lim_{h \to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} \cdot \left(\frac{\frac{1}{\sqrt{x+h}} + \frac{1}{\sqrt{x}}}{\frac{1}{\sqrt{x+h}} + \frac{1}{\sqrt{x}}} \right)$$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h\left(\frac{1}{\sqrt{x+h}} + \frac{1}{\sqrt{x}}\right)}$$

$$= \lim_{h \to 0} \frac{\frac{x - (x+h)}{x}}{h\left(\frac{1}{\sqrt{x+h}} + \frac{1}{\sqrt{x}}\right)}$$

$$= \lim_{h \to 0} \frac{-h}{xh\left(\frac{1}{\sqrt{x+h}} + \frac{1}{\sqrt{x}}\right)} = \lim_{h \to 0} \frac{1}{x\left(\frac{1}{\sqrt{x+h}} + \frac{1}{\sqrt{x}}\right)}$$

$$= \frac{-1}{x \cdot 2\sqrt{x}} = \frac{-1}{2x \cdot x^{\frac{1}{2}}}$$

$$= \frac{-1}{2} \frac{1}{x^{3/2}} = \frac{-1}{2} x^{-3/2}.$$

So, $f'(2)$ = slope of tangent to curve at $x=2$

$$= \frac{-1}{2}(2)^{-3/2}.$$

7. Focal length of lens is fixed, so f is a constant. Then u and v are related by

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v} = f^{-1} = u^{-1} + v^{-1}$$

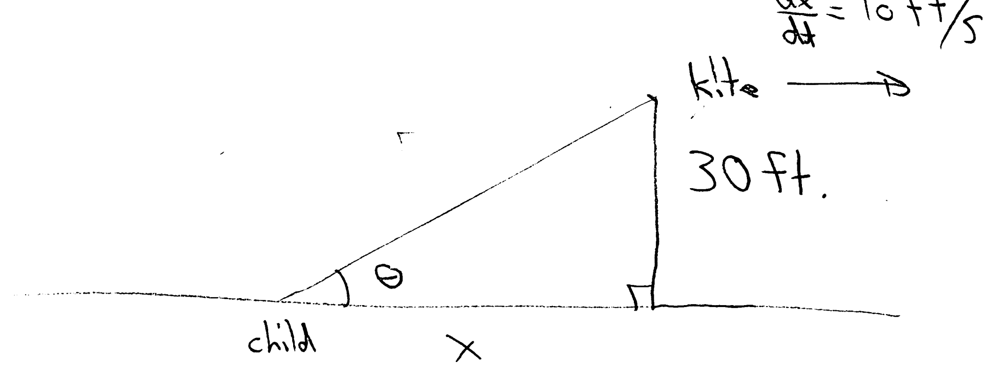
Differentiating, we have

$$0 = -1\, u^{-2} \frac{du}{dv} + -1\, v^{-2}$$

$$\Rightarrow \quad v^{-2} = -u^{-2} \frac{du}{dv}$$

$$\Rightarrow \quad -\frac{v^{-2}}{u^{-2}} = \frac{du}{dv} = -\left(\frac{u}{v}\right)^2 .$$

8.

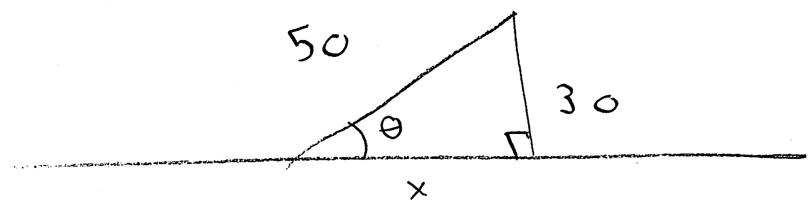


A geometric relation between θ and x is $\tan\theta = \frac{30}{x}$. Differentiating, we get

$$\sec^2\theta \frac{d\theta}{dt} = -30 x^{-2} \frac{dx}{dt}$$

So, $$\frac{d\theta}{dt} = \frac{-30\,(10)}{x^2 \sec^2\theta} .$$

at the moment of interest, 50 ft of string has been let out, so we have this situation:



Using the Pythagorean Theorem, we have

$$x = \sqrt{50^2 - 30^2} = 40, \text{ so}$$

$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{50}{40} = \frac{5}{4}.$$

Plugging into $\frac{d\theta}{dt} = \frac{-30(10)}{x^2 \sec^2\theta}$, we have

$$\frac{d\theta}{dt} = \frac{(-30)(10)}{(40)^2\left(\frac{5}{4}\right)^2}.$$