To all 323 students:

The following are three old midterm exams (titled “Sample 1” “Sample 2” & “Sample 3”) that were given for Math 323 Exam 3. Please ignore the following problems from these exams:

-First Test “Sample 1”: #6

-Second Test “Sample 2”: #4, #8

-Third Test “Sample 3”: #7(a), #8

Math 323 Midterm Examination 3, Sample 1

Problem 1. Evaluate the integral of the function $F(x, y, z) = xyz$ over the part of the unit ball lying in the first octant (that is, all coordinates are non-negative).

a) using spherical coordinates

b) using cylindrical coordinates

Problem 2. Given the curve $\bar{r}(t) =< 2t, t^2, \ln t >$, between points $(2, 1, 0)$ and $(4, 4, \ln 2)$. Evaluate the integral.

$$\int_C f(x, y, z)\ ds$$

where $f(x, y, z) = x - 2y$.

Problem 3. Given the vector field $\bar{F} =< 2xy, x^2 + 4y >$.
a) Determine if it is conservative. If it is, find its potential.

b) Find the integral of $\bar{F}$ over the curve C that connects $(1, 0)$ and $(0, 1)$, going *counter-clockwise* along the unit circle.

Problem 4. Given a vector field $\bar{F} =< 3x^2y, x^3 - 2x >$ and the closed curve C which is a circle of raduis 2 centered at $(3, 2)$, traversed *clockwise*.

a) Set up $\int_C \bar{F} \cdot d\bar{r}$ using the definition of the line integral and a suitable parametrization of C. **Do not calculate.**

b) Find $\int_C \bar{F} \cdot d\bar{r}$ using Green's Theorem.

Problem 5. Find the divergence and the curl of the vector field

$$\bar{F} =< xy, yz, x + z >$$

Problem 6. Find the integral of the function $F(x, y) = x - 3y$ over the parallelogram with vertices $(1, 2)$, $(3, 5)$, $(0, 4)$, and $(-2, 1)$.

Problem 7. Evaluate the integral

$$\iiint\limits_{D} xy^2z \; dV$$

Here D lies above the region on the xy-plane bounded by the parabola $y = x^2$ and the line $y = 4$. And D is bounded above by the sphere of radius 5 centered at the origin.

Problem 8. The curve C is given parametrically as

$$(x, y) = ((10 + \ cos(2018t)) \cos t, (10 + \cos(2018t)) \sin t), \ 0 \leq t \leq 2\pi$$

The vector field $\bar{F}$ is given as $< \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} >$. Find the integral of $\bar{F}$ over C, traversed from $t = 0$ to $t = 2\pi$.

Math 323 Midterm Examination 3, Sample 2

Problem 1. Evaluate the integral $\int_C f(x, y, z)\ ds$, where the curve C is given by

$$\bar{r} =< 3\cos t, 3\sin t, 4t >, \quad 0 \le t \le \pi$$

and $f(x, y, z) = xz$

Problem 2. Given the vector field $\bar{F} =< y^2, y^2 + 2xy >$.
a) Determine if it is conservative. If it is, find its potential.

b) Find the integral of $\bar{F}$ over the straight oriented segment from $A = (2, 0)$ to $B = (0, 1)$.

Problem 3. Find the volume of the solid defined by the inequalities

$$x^2 + y^2 + z^2 \leq 4, \quad z \geq \sqrt{x^2 + y^2}$$

Problem 4. Evaluate the integral of the function $F(x, y) = x^2 + 1$ over the ellipse $4x^2 + y^2 \leq 1$ with respect to area.

Problem 5. Given the vector field

$$\bar{F} =< x^2yz, xy^2z, xyz^2 >$$

a) Find and simplify the curl of $\bar{F}$

b) Find and simplify the divergence of the curl of $\bar{F}$

c) Find and simplify the divergence of $\bar{F}$

d) Find and simplify the gradient of the divergence of $\bar{F}$

Problem 6. Evaluate the integral

$$\iiint\limits_{D} z^2 \, dV$$

where D is the ball of radius 2 centered at the origin.

Problem 7. Evaluate the integral of $\bar{F} =< 2x, xy >$ over the boundary of the square $ABCD$, where $A = (0,0)$, $B = (3,0)$, $C = (3,3)$, and $D = (0,3)$, traversed counter-clockwise.

Problem 8. Construct an example of a vector field $\bar{F} =< P, Q >$ in some domain in the plane, such that $P_y = Q_x$ but $\bar{F}$ is **not** conservative. Justify.

Math 323 Midterm Examination 3, Sample 3

Problem 1. Find the volume of the solid that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and above the paraboloid $z = x^2 + y^2$.

Problem 2. Given the vector field $\bar{F} =< y^2, x >$.
a) Determine if it is conservative. If it is, find its potential.

b) Find the integral of $\bar{F}$ over the straight oriented segment from $A = (1, 0)$ to $B = (0, 1)$.

Problem 3. Evaluate the integral of the function $F(x, y) = x$ over the domain D defined by the inequality $x^2 + 2x + 4y^2 \leq 3$.

Problem 4. Evaluate the integral $\int\limits_C f(x, y, z)\ ds$, where the curve C is given by

$$\bar{r} = < t, t^2, t^3 >, \quad 0 \leq t \leq 1$$

and $f(x, y, z) = 2x + 9z$

Problem 5. Evaluate the integral

$$\iiint\limits_D (x^2 + yz^{2018})\ dV$$

where D is the ball of radius 3 centered at the origin.

Problem 6. Evaluate the integral of $\bar{F} =< xy, xy >$ over the boundary of the square $ABCD$, where $A = (1, 1)$, $B = (1, 2)$, $C = (2, 2)$, and $D = (2, 1)$ (traversed clockwise).

Problem 7. Find the conservative vector field $\bar{F} =< P, Q, R >$ such that

$$P = 2xz,\ Q = 2y + 3z,\ R(0,0,0) = 1,\ R \text{ does not depend on } z.$$

b) Find a potential of $\bar{F}$ from part (a).

c) Find the integral of $\bar{F}$ from part (a) over the curve $\bar{r}(t) =< t, t^2, t^3 >$, where t goes from 0 to 2.

Problem 8. Construct an example of a non-constant vector field $\bar{F}$ on $\mathbb{R}^3$ such that both the divergence and the curl of $\bar{F}$ are identically zero. Justify.

Math 323 Midterm Examination 3, Sample 1

Problem 1. Evaluate the integral of the function $F(x, y, z) = xyz$ over the part of the unit ball lying in the first octant (that is, all coordinates are non-negative).

a) using spherical coordinates

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 (\rho\sin\varphi\cos\theta)(\rho\sin\varphi\sin\theta)(\rho\cos\varphi)\cdot\rho^2\sin\varphi\, d\rho\, d\theta\, d\varphi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \cos\theta\sin\theta\sin^3\varphi\cos\varphi\,\rho^5\, d\rho\, d\theta\, d\varphi = \int_0^{\pi/2}\int_0^{\pi/2} \cos\theta\sin\theta\sin^3\varphi\cos\varphi\cdot\frac{\rho^6}{6}\Big|_0^1 d\theta\, d\varphi$$

$$= \int_0^{\pi/2} \frac{1}{6}\cos\theta\sin\theta\cdot\frac{1}{4}\sin^4\varphi\Big|_0^{\pi/2} d\theta = \int_0^{\pi/2}\frac{1}{24}\sin^2\theta\, d\theta = \boxed{\frac{1}{48}}$$

b) using cylindrical coordinates

$$\int_0^{\pi/2}\int_0^1\int_0^{\sqrt{1-r^2}} r\cos\theta\, r\sin\theta\, z\cdot r\, dz\, dr\, d\theta = \int_0^{\pi/2}\int_0^1 r^3\cos\theta\sin\theta\,\frac{z^2}{2}\Big|_0^{\sqrt{1-r^2}} dr\, d\theta$$

$$= \int_0^{\pi/2}\int_0^1 \cos\theta\sin\theta\cdot\frac{r^3(1-r^2)}{2}\, dr\, d\theta = \int_0^{\pi/2}\cos\theta\sin\theta\cdot\left(\frac{r^4}{8}-\frac{r^6}{12}\right)\Big|_0^1 d\theta$$

$$= \int_0^{\pi/2}\cos\theta\sin\theta\cdot\frac{1}{24}\, d\theta = \frac{1}{48}\sin^2\theta\Big|_0^{\pi/2} = \boxed{\frac{1}{48}}$$

Problem 2. Given the curve $\bar{r}(t) =< 2t, t^2, \ln t >$, between points $(2,1,0)$ and $(4,4,\ln 2)$. Evaluate the integral.

$$\int_C f(x,y,z)\, ds$$

where $f(x,y,z) = x - 2y$.

$\frac{d\vec{r}}{dt} = \langle 2, 2t, \frac{1}{t} \rangle$

$$t \to \int_1^2 (2t-2t^2)\cdot\sqrt{4+4t^2+\frac{1}{t^2}}\,dt$$

$$= \int_1^2 (2t-2t^2)\cdot\left(2t+\frac{1}{t}\right)dt = \int_1^2 (-4t^3+4t^2-2t+2)\,dt$$

$$= \left(-t^4+\frac{4}{3}t^3-t^2+2t\right)\Big|_1^2 = \left(-16+\frac{32}{3}-4+4\right)-\left(-1+\frac{4}{3}-1+2\right)$$

$$= -\frac{16}{3}-\frac{4}{3} = \boxed{-\frac{20}{3}}$$

Problem 3. Given the vector field $\bar{F} =< 2xy, x^2+4y >$.

a) Determine if it is conservative. If it is, find its potential.

$\frac{\partial}{\partial y}(2xy) = 2x = \frac{\partial}{\partial x}(x^2+4y)$ So likely conservative

Suppose $\vec{F} = \nabla P$ $\quad P = \int 2xy\,dx = x^2y + C(y)$

$x^2+4y = P_y = x^2 + C'(y)$. So $C'(y) = 4y$, $C(y) = 2y^2 + \text{Const}$,

$$\boxed{P = x^2y + 2y^2 + c}$$

b) Find the integral of $\bar{F}$ over the curve C that connects $(1,0)$ and $(0,1)$, going *counter-clockwise* along the unit circle.

By FTC, $\int_C \vec{F}\cdot d\vec{r} = P(0,1) - P(1,0) = 2 - 0 = \boxed{2}$

Problem 4. Given a vector field $\bar{F} =< 3x^2y, x^3 - 2x >$ and the closed curve C which is a circle of raduis 2 centered at $(3, 2)$, traversed *clockwise.*

a) Set up $\int_C \bar{F} \cdot d\bar{r}$ using the definition of the line integral and a suitable parametrization of C. **Do not calculate.**

$$\begin{cases} x = 3 + 2\cos t \\ y = 2 - 2\sin t \end{cases} \quad \frac{d\vec{r}}{dt} = \langle -2\sin t, -2\cos t \rangle$$

$$\int_0^{2\pi} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left(3(3+2\cos t)^2(2+2\sin t) \cdot (-2\sin t) + \left((3+2\cos t)^3 - 2(3+2\cos t)\right)(-2\cos t)\right) dt$$

b) Find $\int_C \bar{F} \cdot d\bar{r}$ using Green's Theorem.

$\vec{F} = \langle Q, R \rangle$

orientation

$$\int_C \vec{F} \cdot d\vec{r} = -\iint_D (R_x - Q_y)\, dA = -\iint_D (3x^2 - 2 - 3x^2)\, dA = \iint_D 2\, dA$$

Area of the circle is $\pi \cdot 2^2 = 4\pi$. So $\int_C \vec{F} dr = \boxed{8\pi}$

Problem 5. Find the divergence and the curl of the vector field

$$\bar{F} =< xy, yz, x + z >$$

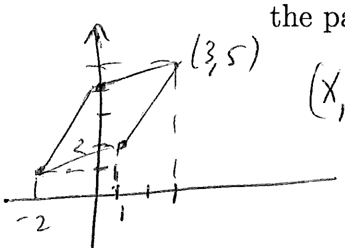
$\text{div}(\vec{F}) = \nabla \cdot \vec{F} = y + z + 1$

$\text{Curl}(\vec{F}) = \nabla \times \vec{F}$

$$\langle \frac{\partial}{\partial x}\ \frac{\partial}{\partial y}\ \frac{\partial}{\partial z} \rangle$$
$$\times \quad \langle xy\ \ yz\ \ x+z \rangle$$

$\text{curl}(\vec{F}) = \langle -y, -1, -x \rangle$

Problem 6. Find the integral of the function $F(x, y) = x - 3y$ over the parallelogram with vertices $(1, 2)$, $(3, 5)$, $(0, 4)$, and $(-2, 1)$.



$(x, y) = (-2, 1) + s(3, 1) + t(2, 3), \quad 0 \le s, t \le 1$

$J = \begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} = 7$

$\begin{cases} x = -2 + 3s + 2t \\ y = 1 + s + 3t \end{cases}$

$\int_0^1 \int_0^1 7 \cdot (-2 + 3s + 2t - 3(1 + s + 3t))\, ds\, dt$

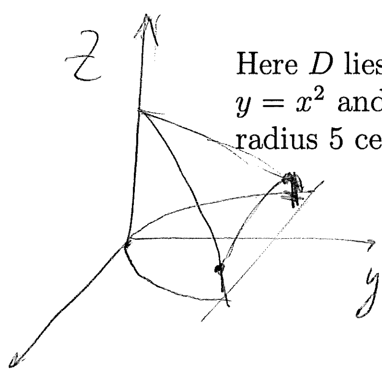
$= 7 \int_0^1 \int_0^1 (-5 - 7t)\, ds\, dt = 7 \int_0^1 (-5 - 7t)\, dt = 7 \left(-5t - \frac{7}{2} t^2\right)\Big|_0^1$

$= 7 \cdot \left(-\frac{17}{2} - 0\right) = \boxed{-\frac{119}{2}}$

Problem 7. Evaluate the integral

$$\iiint_D xy^2 z \; dV$$

Here D lies above the region on the xy-plane bounded by the parabola $y = x^2$ and the line $y = 4$. And D is bounded above by the sphere of radius 5 centered at the origin.



$\int_{-2}^{2} \int_{x^2}^{4} \int_{0}^{\sqrt{25 - x^2 - y^2}} x y^2 z \, dz\, dy\, dx$

$= \int_{-2}^{2} \int_{x^2}^{4} x y^2 \frac{z^2}{2} \Big|_0^{\sqrt{25 - x^2 - y^2}} dy\, dx = \int_{-2}^{2} \int_{x^2}^{4} x y^2 \cdot (25 - x^2 - y^2)\, dy\, dx$

$= \int_{-2}^{2} \left(x(25 - x^2) \frac{y^3}{3} - x \frac{y^5}{5} \right) \Big|_{x^2}^{4} dx = \int_{-2}^{2} \left(x \cdot (25 - x^2)\left(\frac{64}{3} - \frac{x^2}{3}\right) - x \frac{4^5}{3} + \frac{x^{11}}{3} \right) dx$

$= \int_{-2}^{2} x \cdot \left[(25 - x^2)\left(\frac{64}{3} - \frac{x^2}{3}\right) - \frac{4^5}{3} + \frac{x^{10}}{3} \right] dx = \frac{1}{2} \int_4^4 \left((25 - u)\left(\frac{64}{3} - \frac{4}{3}\right) - \frac{4^5}{3} + \frac{4^5}{3} \right) du$

$u = x^2$

$du = 2x\, dx$

$= \boxed{0}$

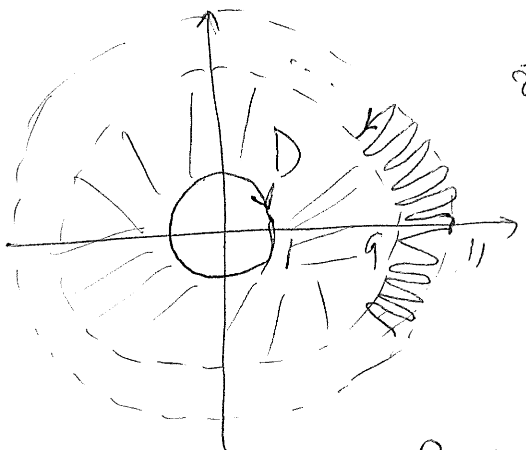
Problem 8. The curve C is given parametrically as

$$(x,y) = ((10 + \; cos(2018t))\cos t, (10+\cos(2018t))\sin t), \;\; 0 \le t \le 2\pi$$

The vector field $\bar{F}$ is given as $< \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} >$. Find the integral of $\bar{F}$ over C, traversed from $t=0$ to $t=2\pi$.

Note that $\vec{F}$ is "locally conservative":

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = 0 \quad \text{(Check!)}$$



Consider the region D bounded by C and by the unit circle. By Green's Theorem,

$$\int_C \vec{F}\cdot d\vec{r} + \int_{\circlearrowright} \vec{F}\cdot d\vec{r} = 0.$$

unit circle

$$\text{So } \int_C \vec{F}\cdot d\vec{r} = \int_{\circlearrowleft} \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \left(\frac{+\sin^2 t}{\cos^2 t+\sin^2 t} + \frac{\cos^2 t}{\cos^2 t+\sin^2 t} \right) dt$$

unit circle

$$= \int_0^{2\pi} 1 \, dt = \boxed{2\pi}$$

$\vec{r} = \langle \cos t, \sin t \rangle$

$\vec{dr} = \langle -\sin t \, dt, \cos t \, dt \rangle$

Math 323 Midterm Examination 3, Sample 2

Problem 1. Evaluate the integral $\int_C f(x,y,z)\,ds$, where the curve C is given by

$$\bar{r} = < 3\cos t, 3\sin t, 4t >, \quad 0 \le t \le \pi$$

and $f(x,y,z) = xz$

$\frac{d\vec{r}}{dt} = \langle -3\sin t, 3\cos t, 4 \rangle$, $\left|\frac{dr}{dt}\right| = \sqrt{9\sin^2 t + 9\cos^2 t + 16} = 5$

$\int_0^\pi 3\cos t \cdot 4t\, 5\,dt = \int_0^\pi 60t \cos t\,dt = 60t\sin t\Big|_0^\pi - \int_0^\pi 60\sin t\,dt$

u, dv; $v = \sin t$; $du = 60\,dt$

$= 0 + 60\cos t\Big|_0^\pi = \boxed{-120}$

Problem 2. Given the vector field $\bar{F} = < y^2, y^2 + 2xy >$.

a) Determine if it is conservative. If it is, find its potential.

$\frac{\partial}{\partial x}(y^2 + 2xy) = 2y$

$\frac{\partial}{\partial y}(y^2) = 2y$ // ← so likely conservative

$F = \nabla P \quad P = \int y^2\,dx = xy^2 + C(y)$

$y^2 + 2xy = P_y = 2xy + C'(y)$, so $C'(y) = y^2$, $C(y) = \frac{1}{3}y^3 + C$

$$\boxed{P = xy^2 + \frac{1}{3}y^3 + C}$$

b) Find the integral of $\bar{F}$ over the straight oriented segment from $A = (2,0)$ to $B = (0,1)$.

By the FTC, $\int_C \vec{F}\,d\vec{r} = P(B) - P(A) = \frac{1}{3} - 0 = \boxed{\frac{1}{3}}$

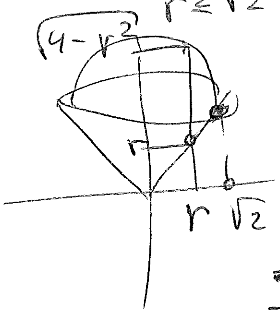
Problem 3. Find the volume of the solid defined by the inequalities

$$x^2 + y^2 + z^2 \le 4, \quad z \ge \sqrt{x^2 + y^2}$$

$r \le z \le \sqrt{4-r^2}$

So $r \le \sqrt{4-r^2}$, $r^2 \le 4 - r^2$, $2r^2 \le 4$, $r \le \sqrt{2}$

$$V = \int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{4-r^2}} 1 \cdot r \, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^{\sqrt{2}} r(\sqrt{4-r^2} - r)\, dr\, d\theta$$



$$= 2\pi \int_0^{\sqrt{2}} (r \cdot \sqrt{4-r^2} - r^2)\, dr = 2\pi\left(-\frac{1}{3}(4-r^2)^{3/2} - \frac{1}{3}r^3\right)\Big|_0^{\sqrt{2}}$$

$$= 2\pi\left(-\frac{1}{3}2^{3/2} - \frac{1}{3}2^{3/2} + \frac{1}{3}\cdot 4^{3/2} - 0\right) =$$

$$= \frac{2\pi}{3}\left(-2\cdot 2^{3/2} + 8\right) = \boxed{\frac{8\pi}{3}(2-\sqrt{2})}$$

Problem 4. Evaluate the integral of the function $F(x, y) = x^2 + 1$ over the ellipse $4x^2 + y^2 \le 1$ with respect to area.

[illegible]

$\begin{cases} u = 2x \\ v = y \end{cases}$, So $\begin{cases} x = \frac{1}{2}u \\ y = v \end{cases}$ $y = \begin{vmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{vmatrix} = \frac{1}{2}$

$$\iint_{u^2+v^2\le 1} \left(\frac{u^2}{4} + 1\right) du = \int_0^{2\pi}\int_0^1 \left(\frac{r^2\cos^2\theta}{4} + 1\right) r\, dr\, d\theta$$

$\begin{cases} u = r\cos\theta \\ v = r\sin\theta \end{cases}$

$$= \int_0^{2\pi}\left(\frac{1}{4}\cos^2\theta \cdot \frac{1}{4}r^4 + \frac{r^2}{2}\right)\Big|_0^1 d\theta = \int_0^{2\pi}\left(\frac{\cos^2\theta}{16} + \frac{1}{2}\right) d\theta$$

$$= \int_0^{2\pi}\left(\frac{1}{32} + \frac{1}{2} + \frac{1}{32}\cos(2\theta)\right) d\theta = 2\pi \cdot \frac{17}{32} + \frac{1}{64}\sin 2\theta\Big|_0^{2\pi}$$

$$= \boxed{\frac{17\pi}{16}}$$

Problem 5. Given the vector field

$$\bar{F} = < x^2yz, xy^2z, xyz^2 >$$

a) Find and simplify the curl of $\bar{F}$

$$\times \begin{array}{c} \left\langle \frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z} \right\rangle \\ \langle x^2yz,\ xy^2z,\ xyz^2 \rangle \end{array}$$

$$\text{curl}(\vec{F}) = \langle xz^2 - xy^2,\ x^2y - yz^2,\ y^2z - x^2z \rangle$$

b) Find and simplify the divergence of the curl of $\bar{F}$

$$\text{div}(\text{curl}(\vec{F})) = (z^2 - y^2) + (x^2 - z^2) + (y^2 - x^2) = 0$$

c) Find and simplify the divergence of $\bar{F}$

$$\text{div}(\vec{F}) = 2xyz + 2xyz + 2xyz = 6xyz$$

d) Find and simplify the gradient of the divergence of $\bar{F}$

$$\nabla(\text{div}(\vec{F})) = \langle 6yz,\ 6xz,\ 6xy \rangle$$

Problem 6. Evaluate the integral

$$\iiint_D z^2 \, dV$$

where D is the ball of radius 2 centered at the origin.

In spherical coordinates:

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{2} (\rho\cos\varphi)^2 \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta = \int_0^{2\pi}\int_0^{\pi} \cos^2\varphi \sin\varphi \left.\frac{\rho^5}{5}\right|_0^2 d\varphi\, d\theta$$

$$= \frac{1}{5} 2\pi \int_0^{\pi} \cos^2\varphi \sin\varphi \, d\varphi = \frac{2\pi}{3}\left(-\frac{1}{3}\cos^3\varphi\right)\Big|_0^{\pi} = \boxed{\frac{4\pi}{9}}$$

Problem 7. Evaluate the integral of $\bar{F} =< 2x, xy >$ over the boundary of the square $ABCD$, where $A = (0,0)$, $B = (3,0)$, $C = (3,3)$, and $D = (0,3)$, traversed counter-clockwise.

By Green's Theorem, $\int_C \vec{F}\cdot d\vec{r} = \iint_D (y - 0)\, dA$

$$= \int_0^3\int_0^3 y \, dx\, dy = \int_0^3 3y\, dy = \frac{3}{2}y^2\Big|_0^3 = \boxed{\frac{27}{2}}$$

(A direct calculation is also possible)

Problem 8. Construct an example of a vector field $\bar{F} = < P, Q >$ in some domain in the plane, such that $P_y = Q_x$ but $\bar{F}$ is **not** conservative. Justify.

The classical example is $\vec{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$,

$D = \mathbb{R}^2 \setminus \{(0,0)\}$.

$$Q_x = \frac{1\cdot(x^2+y^2) - x\cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$P_y = \frac{-(x^2+y^2) + y\cdot 2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)}$$

However $\oint_{\text{unit circle}} \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \left\langle \frac{-\sin t}{1}, \frac{\cos t}{1} \right\rangle \cdot \langle -\sin t, \cos t \rangle dt$

$$= \int_0^{2\pi} 1 \cdot dt = 2\pi.$$

$\begin{cases} x = \cos t \\ y = \sin t \end{cases}$ $d\vec{r} = \langle -\sin t, \cos t \rangle dt$

Because $2\pi \neq 0$, the field $\vec{F}$ is not conservative.

Math 323 Midterm Examination 3, Sample 3

Problem 1. Find the volume of the solid that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and above the paraboloid $z = x^2 + y^2$.

$r^2 \le z \le \sqrt{2-r^2}$; $\quad r^2 = \sqrt{2-r^2} \Rightarrow r^4 + r^2 - 2 = 0$, $\quad r^2 = 1, -2$, $\quad r = 1$ So $0 \le r \le 1$

$$V = \int_0^{2\pi}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} 1 \cdot r\,dz\,dr\,d\theta = \int_0^1\int_0^{2\pi}\left(r\sqrt{2-r^2} - r^3\right)d\theta\,dr = 2\pi\int_0^1\left(r\sqrt{2-r^2} - r^3\right)dr$$

$$= 2\pi\left[-\frac{1}{3}(2-r^2)^{3/2} - \frac{1}{4}r^4\right]\Big|_0^1 = 2\pi\left(\left(-\frac{1}{3} - \frac{1}{4}\right) + \frac{2^{3/2}}{3}\right) = \pi\left(-\frac{7}{6} + \frac{2\cdot 2^{3/2}}{3}\right)$$

$$V = \pi\,\frac{8\sqrt{2} - 7}{6}$$

Problem 2. Given the vector field $\bar{F} = < y^2, x >$.

a) Determine if it is conservative. If it is, find its potential.

$\vec{F} = \langle P, Q \rangle$

$Q_x = 1$

$P_y = 0$ (≠)

So $\vec{F}$ is not conservative, it does not have a potential

b) Find the integral of $\bar{F}$ over the straight oriented segment from $A = (1, 0)$ to $B = (0, 1)$.

$$\begin{cases} x = 1 - t \\ y = t \end{cases} \quad 0 \le t \le 1 \qquad d\vec{r} = \langle -1, 1 \rangle dt$$

$$\int_A^B \vec{F}\cdot d\vec{r} = \int_0^1 \langle t^2, 1-t \rangle \cdot \langle -1, 1 \rangle dt = \int_0^1 \left(-t^2 + 1 - t\right)dt$$

$$= \left(-\frac{1}{3}t^3 + t - \frac{1}{2}t^2\right)\Big|_0^1 = \frac{1}{6}$$

Problem 3. Evaluate the integral of the function $F(x, y) = x$ over the domain D defined by the inequality $x^2 + 2x + 4y^2 \leq 3$.

$(x+1)^2 + 4y^2 \leq 4$ $\qquad$ $(x+1)^2 + (2y)^2 \leq 4$

$\begin{cases} x = u - 1 \\ y = \frac{v}{2} \end{cases}$ $\quad J = \begin{vmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$

$\frac{1}{2} \iint_{u^2+v^2 \leq 4} (u-1)\, du\, dv = \frac{1}{2} \int_0^{2\pi} \int_0^2 (r\cos\theta - 1)\, r\, dr\, d\theta = \frac{1}{2} \int_0^{2\pi} \left(\frac{8}{3}\cos\theta - 2\right) d\theta$

$\begin{cases} u = r\cos\theta \\ v = r\sin\theta \end{cases}$ $\quad = \left(\frac{4}{3}\sin\theta - \theta\right)\Big|_0^{2\pi} = \boxed{-2\pi}$

Problem 4. Evaluate the integral $\int_C f(x, y, z)\, ds$, where the curve C is given by

$$\bar{r} = < t, t^2, t^3 >, \quad 0 \leq t \leq 1$$

and $f(x, y, z) = 2x + 9z$

$\frac{d\vec{r}}{dt} = \langle 1, 2t, 3t^2 \rangle$

$\int_0^1 (2t + 9t^3)\sqrt{1 + 4t^2 + 9t^4}\, dt = \int_1^{14} \frac{1}{4}\sqrt{u}\, du$

$u = 1 + 4t^2 + 9t^4$

$du = 8t + 36t^3$ $\qquad = \frac{1}{4} \cdot \frac{2}{3} u^{3/2} \Big|_1^{14}$

$= \boxed{\frac{1}{6}\left(14^{3/2} - 1\right)}$

Problem 5. Evaluate the integral

$$\iiint_D (x^2 + yz^{2018})\, dV$$

where D is the ball of radius 3 centered at the origin.

Spherical:

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{3}\left(\rho^2\sin^2\varphi\cos^2\theta + \rho^{2019}\sin\varphi\sin\theta\cos^{2018}\varphi\right)\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$\int_0^{2\pi}\left(\tfrac{1}{2}+\tfrac{1}{2}\cos(2\theta)\right) = \left(\tfrac{1}{2}\theta + \tfrac{1}{4}\sin(2\theta)\right)\Big|_0^{2\pi} = \pi \qquad -\cos\theta\Big|_0^{2\pi} = 0$$

$$= \int_0^{\pi}\int_0^{3}\int_0^{2\pi}\left(\rho^4\sin^3\varphi\cos^2\theta + \rho^{2021}\sin^2\varphi\cos^{2018}\varphi\sin\theta\right)d\theta\, d\rho\, d\varphi =$$

$$= \int_0^{\pi}\int_0^{3}\rho^4\sin^3\varphi\cdot\pi\, d\rho\, d\varphi = \pi\int_0^{\pi}\frac{3^5}{5}\sin^3\varphi\, d\varphi = \frac{\pi\cdot 3^5}{5}\int_0^{\pi}(1-\cos^2\varphi)\sin\varphi\, d\varphi$$

$u = \cos\varphi$, $-d\cos\varphi$

$$= \frac{3^5}{5}\pi\cdot\int_1^{-1}(1-u^2)(-du) = \frac{3^5\pi}{5}\cdot\left(u - \frac{u^3}{3}\right)\Big|_{-1}^{1} = \frac{3^5\pi}{5}\cdot\frac{4}{3} = \boxed{\frac{324\pi}{5}}$$

Problem 6. Evaluate the integral of $\bar{F} =< xy, xy >$ over the boundary of the square $ABCD$, where $A = (1,1)$, $B = (1,2)$, $C = (2,2)$, and $D = (2,1)$ (traversed clockwise). $\vec{F} = \langle P, Q\rangle$

By Green's Theorem:

$$\int_C \vec{F}\cdot d\vec{r} = -\int_1^2\int_1^2 (Q_x - P_y)\, dx\, dy = -\int_1^2\int_1^2 (y - x)\, dx\, dy =$$

(clockwise)

$$= -\int_1^2\int_1^2 y\, dx\, dy + \int_1^2\int_1^2 x\, dx\, dy = \boxed{0}$$

Problem 7. Find the conservative vector field $\bar{F} =< P, Q, R >$ such that

$$P = 2xz, \ Q = 2y + 3z, \ R(0,0,0) = 1, \ R \text{ does not depend on } z.$$

$R = R(x, y)$

$\begin{cases} R_x = P_z = 2x \\ R_y = Q_z = 3 \end{cases}$ $\Rightarrow R = x^2 + C(y)$

$C'(y) = 3 \Rightarrow C(y) = 3y + \text{const}$

So $R = x^2 + 3y + \text{const}$

$R(0,0,0) = 1$, So $\boxed{R = x^2 + 3y + 1}$

b) Find a potential of $\bar{F}$ from part (a).

$\vec{F} = \nabla f$, $f_x = 2xz$, $f_y = 2y + 3z$, $f_z = x^2 + 3y + 1$

1) $f = x^2 z + g(y, z)$, $\frac{\partial}{\partial y} g(y,z) = 2y + 3z$

2) $g = y^2 + 3yz + h(z)$, $h'(z) = 1$

3) $h(z) = z + C$, so $\boxed{f = x^2 z + y^2 + 3yz + z + C}$

c) Find the integral of $\bar{F}$ from part (a) over the curve $\bar{r}(t) =< t, t^2, t^3 >$, where t goes from 0 to 2.

$t = 0: \ \vec{r} = \langle 0, 0, 0 \rangle$

$t = 2: \ \vec{r} = \langle 2, 4, 8 \rangle$

By FTC, $\int_C \vec{F} \, d\vec{r} = f(2,4,8) - f(0,0,0)$

$= (\underbrace{4 \cdot 8}_{32} + 16 + \underbrace{3 \cdot 4 \cdot 8}_{96} + 8) - 0 = \boxed{152}$

48   104

Problem 8. Construct an example of a non-constant vector field $\bar{F}$ on $\mathbb{R}^3$ such that both the divergence and the curl of $\bar{F}$ are identically zero. Justify.

Many examples exist:

$\vec{F} = \langle x, y, -2z \rangle$

$\vec{F} = \langle x, -y, 0 \rangle$

$\vec{F} = \langle yz, xz, xy \rangle \quad (= \nabla(xyz))$

...

To justify, just need to calculate $\operatorname{div}(\vec{F})$ and $\operatorname{curl}(\vec{F})$.