To all 323 students:

The following are two old midterm exams (titled "Sample 1" "Sample 2" & "Sample 3") that were given for Math 323 Exam 3. Please ignore the following problems from these exams:

-First Test "Sample 1": #6

-Third Test "Sample 3": #7(a), #8

Math 323 Midterm Examination 3, Sample 1

Problem 1. Evaluate the integral of the function $F(x, y, z) = xyz$ over the part of the unit ball lying in the first octant (that is, all coordinates are non-negative).

a) using spherical coordinates

b) using cylindrical coordinates

Problem 2. Given the curve $\bar{r}(t) = < 2t, t^2, \ln t >$, between points $(2, 1, 0)$ and $(4, 4, \ln 2)$. Evaluate the integral.

$$\int_C f(x, y, z)\ ds$$

where $f(x, y, z) = x - 2y$.

Problem 3. Given the vector field $\bar{F} = < 2xy, x^2 + 4y >$.
a) Determine if it is conservative. If it is, find its potential.

b) Find the integral of $\bar{F}$ over the curve C that connects $(1, 0)$ and $(0, 1)$, going *counter-clockwise* along the unit circle.

Problem 4. Given a vector field $\bar{F} =< 3x^2y, x^3 - 2x >$ and the closed curve C which is a circle of raduis 2 centered at $(3, 2)$, traversed *clockwise*.

a) Set up $\int\limits_{C} \bar{F} \cdot d\bar{r}$ using the definition of the line integral and a suitable parametrization of C. **Do not calculate.**

b) Find $\int\limits_{C} \bar{F} \cdot d\bar{r}$ using Green's Theorem.

Problem 5. Find the divergence and the curl of the vector field

$$\bar{F} =< xy, yz, x + z >$$

Problem 6. Find the integral of the function $F(x, y) = x - 3y$ over the parallelogram with vertices $(1, 2)$, $(3, 5)$, $(0, 4)$, and $(-2, 1)$.

Problem 7. Evaluate the integral

$$\iiint_D xy^2z \ dV$$

Here D lies above the region on the xy-plane bounded by the parabola $y = x^2$ and the line $y = 4$. And D is bounded above by the sphere of radius 5 centered at the origin.

Problem 8. The curve C is given parametrically as

$$(x, y) = ((10 + \cos(2018t)) \cos t, (10 + \cos(2018t)) \sin t), \ 0 \le t \le 2\pi$$

The vector field $\bar{F}$ is given as $< \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} >$. Find the integral of $\bar{F}$ over C, traversed from $t = 0$ to $t = 2\pi$.

Math 323 Midterm Examination 3, Sample 2

Problem 1. Evaluate the integral $\int_C f(x, y, z) \, ds$, where the curve C is given by

$$\bar{r} = < 3\cos t, 3\sin t, 4t >, \quad 0 \le t \le \pi$$

and $f(x, y, z) = xz$

Problem 2. Given the vector field $\bar{F} = < y^2, y^2 + 2xy >$.
a) Determine if it is conservative. If it is, find its potential.

b) Find the integral of $\bar{F}$ over the straight oriented segment from $A = (2, 0)$ to $B = (0, 1)$.

Problem 3. Find the volume of the solid defined by the inequalities

$$x^2 + y^2 + z^2 \leq 4, \quad z \geq \sqrt{x^2 + y^2}$$

Problem 4. Evaluate the integral of the function $F(x, y) = x^2 + 1$ over the ellipse $4x^2 + y^2 \leq 1$ with respect to area.

Problem 5. Given the vector field
$$\bar{F} =< x^2yz, xy^2z, xyz^2 >$$
a) Find and simplify the curl of $\bar{F}$

b) Find and simplify the divergence of the curl of $\bar{F}$

c) Find and simplify the divergence of $\bar{F}$

d) Find and simplify the gradient of the divergence of $\bar{F}$

Problem 6. Evaluate the integral

$$\iiint_D z^2 \, dV$$

where D is the ball of radius 2 centered at the origin.

Problem 7. Evaluate the integral of $\bar{F} = < 2x, xy >$ over the boundary of the square $ABCD$, where $A = (0,0)$, $B = (3,0)$, $C = (3,3)$, and $D = (0,3)$, traversed counter-clockwise.

Problem 8. Construct an example of a vector field $\bar{F} = <P, Q>$ in some domain in the plane, such that $P_y = Q_x$ but $\bar{F}$ is **not** conservative. Justify.

Math 323 Midterm Examination 3, Sample 3

Problem 1. Find the volume of the solid that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and above the paraboloid $z = x^2 + y^2$.

Problem 2. Given the vector field $\bar{F} =< y^2, x >$.
a) Determine if it is conservative. If it is, find its potential.

b) Find the integral of $\bar{F}$ over the straight oriented segment from $A = (1, 0)$ to $B = (0, 1)$.

Problem 3. Evaluate the integral of the function $F(x, y) = x$ over the domain D defined by the inequality $x^2 + 2x + 4y^2 \leq 3$.

Problem 4. Evaluate the integral $\int\limits_C f(x, y, z) \; ds$, where the curve C is given by
$$\bar{r} = < t, t^2, t^3 >, \quad 0 \leq t \leq 1$$
and $f(x, y, z) = 2x + 9z$

Problem 5. Evaluate the integral

$$\iiint_D (x^2 + yz^{2018}) \, dV$$

where D is the ball of radius 3 centered at the origin.

Problem 6. Evaluate the integral of $\bar{F} =< xy, xy >$ over the boundary of the square $ABCD$, where $A = (1,1)$, $B = (1,2)$, $C = (2,2)$, and $D = (2,1)$ (traversed clockwise).

Problem 7. Find the conservative vector field $\bar{F} =< P, Q, R >$ such that

$$P = 2xz, \ Q = 2y + 3z, \ R(0,0,0) = 1, \ R \text{ does not depend on } z.$$

b) Find a potential of $\bar{F}$ from part (a).

c) Find the integral of $\bar{F}$ from part (a) over the curve $\bar{r}(t) =< t, t^2, t^3 >$, where t goes from 0 to 2.

Problem 8. Construct an example of a non-constant vector field $\bar{F}$ on $\mathbb{R}^3$ such that both the divergence and the curl of $\bar{F}$ are identically zero. Justify.

Problem 1. Evaluate the integral of the function $F(x, y, z) = xyz$ over the part of the unit ball lying in the first octant (that is, all coordinates are non-negative).

a) using spherical coordinates

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 (\rho\sin\varphi\cos\theta)(\rho\sin\varphi\sin\theta)(\rho\cos\varphi)\cdot\rho^2\sin\varphi \; d\rho \, d\theta \, d\varphi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \cos\theta\sin\theta\sin^3\varphi\cos\varphi\,\rho^5\,d\rho\,d\theta\,d\varphi = \int_0^{\pi/2}\int_0^{\pi/2}\cos\theta\sin\theta\sin^3\varphi\cos\varphi\cdot\frac{\rho^6}{6}\Big|_0^1 d\theta\,d\varphi$$

$$= \int_0^{\pi/2}\frac{1}{6}\cos\theta\sin\theta\cdot\frac{1}{4}\sin^4\varphi\Big|_0^{\pi/2} d\theta = \int_0^{\pi/2}\frac{1}{24}\sin^2\theta\,d\theta = \boxed{\frac{1}{48}}$$

b) using cylindrical coordinates

$$\int_0^{\pi/2}\int_0^1\int_0^{\sqrt{1-r^2}} r\cos\theta\, r\sin\theta\, z\cdot r\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^1 r^3\cos\theta\sin\theta\,\frac{z^2}{2}\Big|_0^{\sqrt{1-r^2}} dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^1 \cos\theta\sin\theta\cdot\frac{r^3(1-r^2)}{2}dr\,d\theta = \int_0^{\pi/2}\cos\theta\sin\theta\cdot\left(\frac{r^4}{8}-\frac{r^6}{12}\right)\Big|_0^1 d\theta$$

$$= \int_0^{\pi/2}\cos\theta\sin\theta\cdot\frac{1}{24}d\theta = \frac{1}{48}\sin^2\theta\Big|_0^{\pi/2} = \boxed{\frac{1}{48}}$$

Problem 2. Given the curve $\bar{r}(t) = <2t, t^2, \ln t>$, between points $(2, 1, 0)$ and $(4, 4, \ln 2)$. Evaluate the integral.

$$\int_C f(x, y, z)\, ds$$

where $f(x, y, z) = x - 2y$.

$$\frac{d\vec{r}}{dt} = <2, 2t, \frac{1}{t}>$$

$$t \nearrow \searrow \int_1^2 (2t - 2t^2) \cdot \sqrt{4 + 4t^2 + \frac{1}{t^2}}\, dt$$

$$= \int_1^2 (2t - 2t^2) \cdot (2t + \frac{1}{t})\, dt = \int_1^2 (-4t^3 + 4t^2 - 2t + 2)\, dt$$

$$= (-t^4 + \frac{4}{3}t^3 - t^2 + 2t)\Big|_1^2 = (-16 + \frac{32}{3} - 4 + 4) - (-1 + \frac{4}{3} - 1 + 2)$$

$$= -\frac{16}{3} - \frac{4}{3} = \boxed{-\frac{20}{3}}$$

Problem 3. Given the vector field $\bar{F} = <2xy, x^2 + 4y>$.
a) Determine if it is conservative. If it is, find its potential.

$$\frac{\partial}{\partial y}(2xy) = 2x = \frac{\partial}{\partial x}(x^2 + 4y) \quad \text{So likely conservative}$$

$$\text{Suppose } \bar{F} = \nabla P \qquad P = \int 2xy\, dx = x^2 y + C(y)$$

$$x^2 + 4y = P_y = x^2 + C'(y). \quad \text{So } C'(y) = 4y, \quad C(y) = 2y^2 + \text{Const,}$$

$$\boxed{P = x^2 y + 2y^2 + C}$$

b) Find the integral of $\bar{F}$ over the curve C that connects $(1, 0)$ and $(0, 1)$, going *counter-clockwise* along the unit circle.

$$\text{By FTC,} \quad \int_C \vec{F} \cdot d\vec{r} = P(0, 1) - P(1, 0) = 2 - 0 = \boxed{2}$$

Problem 4. Given a vector field $\bar{F} = <3x^2y, x^3 - 2x>$ and the closed curve C which is a circle of raduis 2 centered at $(3,2)$, traversed *clockwise.*

a) Set up $\int_C \bar{F} \cdot d\bar{r}$ using the definition of the line integral and a suitable parametrization of C. **Do not calculate.**

$$\begin{cases} x = 3 + 2\cos t \\ y = 2 + 2\sin t \end{cases} \qquad \frac{d\vec{r}}{dt} = <-2\sin t, -2\cos t>$$

$$\int_0^{2\pi} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left(3(3+2\cos t)^2 (2+2\sin t) \cdot (-2\sin t) + \left[(3+2\cos t)^3 - 2(3+2\cos t) \right] (-2\cos t) \right) dt$$

b) Find $\int_C \bar{F} \cdot d\bar{r}$ using Green's Theorem.

$$\vec{F} = <Q, R>$$

orientation

$$\int_C \vec{F} \cdot d\vec{r} = - \iint_D (R_x - Q_y) \, dA = -\iint_D (3x^2 - 2 - 3x^2) \, dA = \iint_D 2 \, dA$$

Area of the circle is $\pi \cdot 2^2 = 4\pi$. So $\int_C \vec{F} \, dr = \boxed{8\pi}$

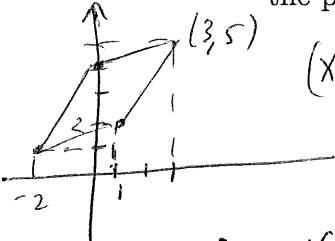**Problem 5.** Find the divergence and the curl of the vector field
$$\bar{F} = <xy, yz, x+z>$$

$$\text{div}(\vec{F}) = \nabla \cdot \vec{F} = y + z + 1$$

$$\text{curl}(\vec{F}) = \nabla \times \vec{F} \qquad \begin{vmatrix} & <\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}> \\ x & <xy & yz & x+z> \end{vmatrix}$$

$$\text{curl}(\vec{F}) = <-y, -1, -x>$$

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Problem 6. Find the integral of the function $F(x,y) = x - 3y$ over the parallelogram with vertices $(1,2)$, $(3,5)$, $(0,4)$, and $(-2,1)$.

$(x,y) = (-2,1) + s(3,1) + t(2,3), \qquad 0 \le s, t \le 1$

$$J = \begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} = 7$$

$\begin{cases} x = -2 + 3s + 2t \\ y = 1 + s + 3t \end{cases}$

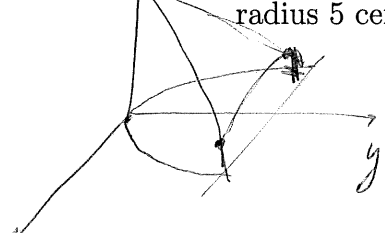$$\int_0^1 \int_0^1 7 \cdot \left(-2 + 3s + 2t - 3(1 + s + 3t) \right) ds\, dt$$

$$= 7 \int_0^1 \int_0^1 (-5 - 7t)\, ds\, dt = 7 \int_0^1 (-5 - 7t)\, dt = 7 \left(-5t - \frac{7}{2}t^2 \right) \Big|_0^1$$

$$= 7 \cdot \left(-\frac{17}{2} - 0 \right) = \boxed{-\frac{119}{2}}$$

Problem 7. Evaluate the integral

$$\iiint_D xy^2 z \, dV$$

Here D lies above the region on the xy-plane bounded by the parabola $y = x^2$ and the line $y = 4$. And D is bounded above by the sphere of radius 5 centered at the origin.

$$\int_{-2}^2 \int_{x^2}^4 \int_0^{\sqrt{25 - x^2 - y^2}} x y^2 z \, dz\, dy\, dx$$

$$= \int_{-2}^2 \int_{x^2}^4 x y^2 \frac{z^2}{2} \Big|_0^{\sqrt{25-x^2-y^2}} dy\, dx = \int_{-2}^2 \int_{x^2}^4 xy^2 \cdot (25 - x^2 - y^2)\, dy\, dx$$

$$= \int_{-2}^2 \left(x(25 - x^2) \frac{y^3}{3} - x \frac{y^5}{5} \right) \Big|_{x^2}^4 dx = \int_{-2}^2 \left(x \cdot (25 - x^2)\left(\frac{64}{3} - \frac{x^2}{3} \right) - x \frac{4^5}{3} + x^{11} \cdot \frac{1}{3} \right) dx$$

$$= \int_{-2}^2 x \cdot \left[(25 - x^2)\left(\frac{64}{3} - \frac{x^2}{3} \right) - \frac{4^5}{3} + \frac{x^{10}}{3} \right] dx = \frac{1}{2} \int_4^4 (25 - u)\left(\frac{64}{3} - \frac{4}{3} \right) - \frac{4^5}{3} + \frac{4^5}{3}\, du$$
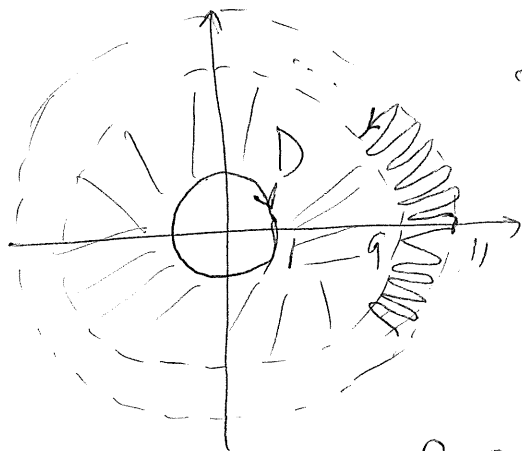
$$u = x^2 \qquad = \boxed{0}$$

$du = 2x\, dx$

Problem 8. The curve C is given parametrically as

$(x, y) = ((10 + \cos(2018t))\cos t, (10 + \cos(2018t))\sin t), \ 0 \le t \le 2\pi$

The vector field $\bar{F}$ is given as $< \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} >$. Find the integral of $\bar{F}$ over C, traversed from $t = 0$ to $t = 2\pi$.

Note that $\vec{F}$ is "locally conservative":

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = 0 \quad \text{(Check!)}$$

Consider the region D bounded by C and by the unit circle. By Green's Theorem,

$$\int_C \vec{F} \cdot d\vec{r} + \int_{\circlearrowleft} \vec{F} \cdot d\vec{r} = 0.$$

So $\displaystyle\int_C \vec{F} \cdot d\vec{r} = \int_{\circlearrowleft} \vec{F} \cdot d\vec{r} = \int_0^{2\pi}\left(\frac{+\sin^2 t}{\cos^2 t + \sin^2 t} + \frac{\cos^2 t}{\cos^2 t + \sin^2 t}\right)dt$

unit circle

$$= \int_0^{2\pi} 1 \, dt = \boxed{2\pi}$$

$\vec{r} = <\cos t, \sin t>$

$d\vec{r} = <-\sin t \, dt, \cos t \, dt>$

Math 323 Midterm Examination 3, Sample 2

Problem 1. Evaluate the integral $\int_C f(x,y,z)\, ds$, where the curve C is given by

$$\bar r = <3\cos t, 3\sin t, 4t>, \quad 0 \le t \le \pi$$

and $f(x,y,z) = xz$

$$\frac{d\bar r}{dt} = <-3\sin t, 3\cos t, 4>, \quad \left|\frac{d\bar r}{dt}\right| = \sqrt{9\sin^2 t + 9\cos^2 t + 16} = 5$$

$$\int_0^\pi 3\cos t \cdot 4t \cdot 5\, dt = \int_0^\pi \underbrace{60t}_{u}\underbrace{\cos t\, dt}_{dv} = 60t\sin t\Big|_0^\pi - \int_0^\pi 60\sin t\, dt$$

$$v = \sin t$$
$$du = 60\, dt$$

$$= 0 + 60\cos t\Big|_0^\pi = \boxed{-120}$$

Problem 2. Given the vector field $\bar F = <y^2, y^2 + 2xy>$.
a) Determine if it is conservative. If it is, find its potential.

$$\frac{\partial}{\partial x}(y^2 + 2xy) = 2y \quad \text{so likely conservative}$$
$$\frac{\partial}{\partial y}(y^2) = 2y \quad // \quad F = \nabla P \quad P = \int y^2 dx = xy^2 + c(y)$$

$$y^2 + 2xy = P_y = 2xy + c'(y), \quad \text{so } c'(y) = y^2, \; c(y) = \frac{1}{3}y^3 + c$$

$$\boxed{P = xy^2 + \frac{1}{3}y^3 + c}$$

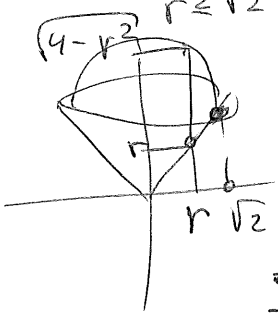b) Find the integral of $\bar F$ over the straight oriented segment from $A = (2,0)$ to $B = (0,1)$.

By the FTC, $\int_C \bar F\, d\bar r = P(B) - P(A) = \frac{1}{3} - 0 = \boxed{\frac{1}{3}}$

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Problem 3. Find the volume of the solid defined by the inequalities

$$x^2 + y^2 + z^2 \le 4, \quad z \ge \sqrt{x^2 + y^2}$$

$$r \le z \le \sqrt{4 - r^2}$$

So $\quad r \le \sqrt{4 - r^2}$,

$$r^2 \le 4 - r^2$$
$$2r^2 \le 4$$
$$r \le \sqrt{2}$$

$$r \le \sqrt{2}$$

$$V = \int_0^{2\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} 1 \cdot r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^{\sqrt{2}} r\left(\sqrt{4-r^2} - r\right) dr \, d\theta$$

$$= 2\pi \int_0^{\sqrt{2}} \left(r\sqrt{4-r^2} - r^2\right) dr = 2\pi\left(-\frac{1}{3}(4-r^2)^{3/2} - \frac{1}{3}r^3\right)\Big|_0^{\sqrt{2}}$$

$$= 2\pi\left(-\frac{1}{3}2^{3/2} - \frac{1}{3}2^{3/2} + \frac{1}{3}\cdot 4^{3/2} - 0\right) =$$

$$= \frac{2\pi}{3}\left(-2\cdot 2^{3/2} + 8\right) = \boxed{\frac{8\pi}{3}\left(2 - \sqrt{2}\right)}$$

Problem 4. Evaluate the integral of the function $F(x,y) = x^2 + 1$
over the ellipse $4x^2 + y^2 \le 1$ with respect to area.

$$\begin{cases} u = 2x \\ v = y \end{cases} \quad \text{So} \quad \begin{cases} x = \frac{1}{2}u \\ y = v \end{cases} \quad y = \begin{vmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{vmatrix} = \frac{1}{2}$$

$$\iint_{u^2+v^2\le 1} \left(\frac{u^2}{4} + 1\right) du = \int_0^{2\pi} \int_0^1 \left(\frac{r^2\cos^2\theta}{4} + 1\right) r \, dr \, d\theta$$

$$\begin{cases} u = r\cos\theta \\ v = r\sin\theta \end{cases}$$

$$= \int_0^{2\pi} \left(\frac{1}{4}\cos^2\theta \cdot \frac{1}{4}r^4 + \frac{r^2}{2}\right)\Big|_0^1 d\theta = \int_0^{2\pi} \left(\frac{\cos^2\theta}{16} + \frac{1}{2}\right) d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{32} + \frac{1}{2} + \frac{1}{32}\cos(2\theta)\right) d\theta = 2\pi\cdot\frac{17}{32} + \frac{1}{64}\sin 2\theta\Big|_0^{2\pi}$$

$$= \boxed{\frac{17\pi}{16}}$$

Problem 5. Given the vector field

$$\bar{F} =< x^2yz, xy^2z, xyz^2 >$$

a) Find and simplify the curl of $\bar{F}$

$$\left< \frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z} \right>$$

$$X$$

$$\frac{< x^2yz, \quad xy^2z, xyz^2 >}{}$$

$$curl(\vec{F}) = \left< xz^2 - xy^2, \; x^2y - yz^2, \; y^2z - x^2z \right>$$

b) Find and simplify the divergence of the curl of $\bar{F}$

$$div\left(curl(\vec{F})\right) = (z^2 - y^2) + (x^2 - z^2) + (y^2 - x^2) = 0$$

c) Find and simplify the divergence of $\bar{F}$

$$div(\vec{F}) = 2xyz + 2xyz + 2xyz = 6xyz$$

d) Find and simplify the gradient of the divergence of $\bar{F}$

$$\nabla\left(div(\vec{F})\right) = \left< 6yz, \; 6xz, \; 6xy \right>$$

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Problem 6. Evaluate the integral

$$\iiint_D z^2 \, dV$$

where D is the ball of radius 2 centered at the origin.

In spherical coordinates:

$$\int_0^{2\pi} \int_0^{\pi} \int_0^2 (\rho \cos\varphi)^2 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = \int_0^{2\pi} \int_0^{\pi} \cos^2\varphi \sin\varphi \left. \frac{\rho^5}{5} \right|_0^2 d\varphi \, d\theta$$

$$= \frac{\boxed{?}}{5} \cdot 2\pi \int_0^{\pi} \cos^2\varphi \sin\varphi \, d\varphi = \frac{2\pi}{3} \left. \left(-\frac{1}{3}\cos^3\varphi \right) \right|_0^{\pi} = \boxed{\frac{4\pi}{9}}$$

Problem 7. Evaluate the integral of $\bar{F} = <2x, xy>$ over the boundary of the square $ABCD$, where $A = (0,0)$, $B = (3,0)$, $C = (3,3)$, and $D = (0,3)$, traversed counter-clockwise.

By Green's Theorem, $\quad \oint \vec{F} \cdot d\vec{r} = \iint (y - 0) \, dA$

$$= \int_0^3 \int_0^3 y \, dx \, dy = \int_0^3 3y \, dy = \left. \frac{3}{2}y^2 \right|_0^3 = \boxed{\frac{27}{2}}$$

(A direct calculation is also possible)

Problem 8. Construct an example of a vector field $\bar{F} =< P, Q >$ in some domain in the plane, such that $P_y = Q_x$ but $\bar{F}$ is **not** conservative. Justify.

The classical example is $\vec{F} = \left\langle \dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right\rangle$,

$$D = \mathbb{R}^2 \setminus \{(0,0)\}.$$

$$Q_x = \frac{1\cdot(x^2+y^2) - x\cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$P_y = \frac{-(x^2+y^2) + y\cdot 2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

However $\displaystyle\oint \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \left\langle \frac{-\sin t}{1}, \frac{\cos t}{1} \right\rangle \cdot \langle -\sin t, \cos t\rangle\, dt$

(unit circle)

$$= \int_0^{2\pi} 1\cdot dt = 2\pi.$$

$\begin{cases} x = \cos t \\ y = \sin t \end{cases}$ $d\vec{r} = \langle -\sin t, \cos t\rangle\, dt$

Because $2\pi \neq 0$, the field $\vec{F}$ is not conservative.

Problem 1. Find the volume of the solid that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and above the paraboloid $z = x^2 + y^2$.

$r^2 \leq z \leq \sqrt{2-r^2}$; $r^2 = \sqrt{2-r^2} \Rightarrow r^4 + r^2 - 2 = 0$, $r^2 = 1, -2$, $\boxed{r = 1}$ So $0 \leq r \leq 1$

$$V = \int_0^{2\pi}\int_0^1 \int_{r^2}^{\sqrt{2-r^2}} 1 \cdot r \, dz \, dr \, d\theta = \int_0^{2\pi}\int_0^1 (r\sqrt{2-r^2} - r^3)\,d\theta \, dr = 2\pi \int_0^1 (r\sqrt{2-r^2} - r^3)\, dr$$

$$= 2\pi\left[\frac{-1}{3}(2-r^2)^{3/2} - \frac{1}{4}r^4\right]\Big|_0^1 = 2\pi\left(\left(-\frac{1}{3} - \frac{1}{4}\right) + \frac{2^{3/2}}{3}\right) = \pi\left(-\frac{7}{6} + \frac{2\cdot 2^{3/2}}{3}\right)$$

$$\boxed{V = \pi \frac{8\sqrt{2}-7}{6}}$$

Problem 2. Given the vector field $\bar{F} = <y^2, x>$.
a) Determine if it is conservative. If it is, find its potential.

$\vec{F} = <P, Q>$

$Q_x = 1$

$P_y = 0$

So $\vec{F}$ is <u>not</u> conservative, it does not have a potential

b) Find the integral of $\bar{F}$ over the straight oriented segment from $A = (1,0)$ to $B = (0,1)$.

$$\begin{cases} x = 1-t \\ y = t \end{cases} \quad 0 \leq t \leq 1 \qquad d\vec{r} = <-1, 1>\, dt$$

$$\int_A^B \vec{F}\cdot d\vec{r} = \int_0^1 <t^2, 1-t>\cdot <-1, 1>\,dt = \int_0^1 (-t^2 + 1-t)\,dt$$

$$= \left(-\frac{1}{3}t^3 + t - \frac{1}{2}t^2\right)\Big|_0^1 = \boxed{\frac{1}{6}}$$

Problem 3. Evaluate the integral of the function $F(x, y) = x$ over the domain D defined by the inequality $x^2 + 2x + 4y^2 \leq 3$.

$$(x+1)^2 + 4y^2 \leq 4 \qquad (x+1)^2 + (2y)^2 \leq 4$$

$$\begin{cases} x = u - 1 \\ y = \frac{v}{2} \end{cases} \quad y = \begin{vmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

$$\frac{1}{2}\iint_{u^2 + v^2 \leq 4} (u-1)\, du\, dv = \frac{1}{2}\int_0^{2\pi}\int_0^2 (r\cos\theta - 1)\, r\, dr\, d\theta = \frac{1}{2}\int_0^{2\pi}\left(\frac{8}{3}\cos\theta - 2\right)d\theta$$

$$\begin{cases} u = r\cos\theta \\ v = r\sin\theta \end{cases} \qquad = \left(\frac{4}{3}\sin\theta - \theta\right)\Big|_0^{2\pi} = \boxed{-2\pi}$$

Problem 4. Evaluate the integral $\int_C f(x, y, z)\, ds$, where the curve C is given by

$$\bar{r} = \langle t, t^2, t^3 \rangle, \quad 0 \leq t \leq 1$$

and $f(x, y, z) = 2x + 9z$

$$\frac{d\bar{r}}{dt} = \langle 1, 2t, 3t^2 \rangle$$

$$\int_0^1 (2t + 9t^3)\sqrt{1 + 4t^2 + 9t^4}\, dt = \int_1^{14} \frac{1}{4}\sqrt{u}\, du$$

$$u = 1 + 4t^2 + 9t^4$$

$$du = 8t + 36t^3$$

$$= \frac{1}{4}\cdot\frac{2}{3}u^{3/2}\Big|_1^{14}$$

$$= \boxed{\frac{1}{6}\left(14^{3/2} - 1\right)}$$

Problem 5. Evaluate the integral

$$\iiint_D (x^2 + yz^{2018}) \, dV$$

where D is the ball of radius 3 centered at the origin.

Spherical:

$$\int_0^{2\pi} \int_0^{\pi} \int_0^3 \left(\rho^2 \sin^2\varphi \cos^2\theta + \rho^{2019} \sin\varphi \sin\theta \cos^{2018}\varphi \right) \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$\int_0^{2\pi}\left(\tfrac{1}{2} + \tfrac{1}{2}\cos(2\theta)\right) = \left(\tfrac{1}{2}\theta + \tfrac{1}{4}\sin(2\theta)\right)\Big|_0^{2\pi} = \pi \qquad -\cos\theta\Big|_0^{2\pi} = 0$$

$$u = \cos\varphi$$
$$-d\cos\varphi$$

$$= \int_0^{\pi} \int_0^{2\pi} \int_0^3 \left(\rho^4 \sin^3\varphi \cos^2\theta + \rho^{2021} \sin^2\varphi \cos^{2018}\varphi \sin\theta \right) d\theta \, d\rho \, d\varphi$$

$$= \int_0^{\pi} \int_0^3 \rho^4 \sin^3\varphi \cdot \pi \, d\rho \, d\varphi = \pi \int \frac{3^5}{5} \sin^3\varphi \, d\varphi = \frac{\pi \cdot 3^5}{5} \int (1 - \cos^2\varphi) \sin\varphi \, d\varphi$$

$$= \frac{3^5}{5} \pi \cdot \int_1^{-1} (1 - u^2)(-du) = \frac{3^5 \pi}{5} \cdot \left(u - \frac{u^3}{3} \right)\Big|_{-1}^{1} = \frac{3^5 \pi}{5} \cdot \frac{4}{3} = \boxed{\dfrac{324\pi}{5}}$$

Problem 6. Evaluate the integral of $\bar{F} =\, <xy, xy>$ over the boundary of the square $ABCD$, where $A = (1,1)$, $B = (1,2)$, $C = (2,2)$, and $D = (2,1)$ (traversed clockwise). $\vec{F} = <P, Q>$

By Green's Theorem:

$$\int_C \vec{F} \cdot d\vec{r} = -\int_1^2 \int_1^2 (Q_x - P_y) \, dx \, dy = -\int_1^2 \int_1^2 (y - x) \, dx \, dy =$$

clockwise

$$= -\int_1^2 \int_1^2 y \, dx \, dy + \int_1^2 \int_1^2 x \, dx \, dy = \boxed{0}$$

Problem 7. Find the conservative vector field $\bar{F} =< P, Q, R >$ such that

$$P = 2xz, \quad Q = 2y + 3z, \quad R(0,0,0) = 1, \quad R \text{ does not depend on } z.$$

$R = R(x, y)$

$\begin{cases} R_x = P_z = 2x \implies R = x^2 + C(y) \\ R_y = Q_z = 3 \end{cases}$

$C'(y) = 3 \implies C(y) = 3y + const$

So $R = x^2 + 3y + const$

$R(0,0,0) = 1$, So $\boxed{R = x^2 + 3y + 1}$

b) Find a potential of $\bar{F}$ from part (a).

$\vec{F} = \nabla f, \quad f_x = 2xz, \quad f_y = 2y + 3z, \quad f_z = x^2 + 3y + 1$

1) $f = x^2 z + g(y, z), \quad \frac{\partial}{\partial y} g(y, z) = 2y + 3z$

2) $g = y^2 + 3yz + h(z), \quad h'(z) = 1$

3) $h(z) = z + C$, So $\boxed{f = x^2 z + y^2 + 3yz + z + C}$

c) Find the integral of $\bar{F}$ from part (a) over the curve $\bar{r}(t) =< t, t^2, t^3 >$, where t goes from 0 to 2.

$t = 0: \quad \vec{r} = \langle 0, 0, 0 \rangle$

$t = 2: \quad \vec{r} = \langle 2, 4, 8 \rangle$

By FTC, $\quad \int_C \vec{F} \, d\vec{r} = f(2, 4, 8) - f(0, 0, 0)$

$= \left(\underbrace{4 \cdot 8 + 16}_{48} + \underbrace{3 \cdot 4 \cdot 8 + 8}_{96} \right) - 0 = \boxed{152}$

$\underbrace{\quad}_{104}$

Problem 8. Construct an example of a non-constant vector field $\bar{F}$ on $\mathbb{R}^3$ such that both the divergence and the curl of $\bar{F}$ are identically zero. Justify.

Many examples exist:

$$\vec{F} = \langle x, y, -2z \rangle$$

$$\vec{F} = \langle x, -y, 0 \rangle$$

$$\vec{F} = \langle yz, xz, xy \rangle \qquad \left(= \nabla(xyz) \right)$$

$\cdot\cdot\cdot$

To justify, just need to calculate $\text{div}(\vec{F})$ and $\text{curl}(\vec{F})$.