Department of Mathematical Sciences

Math 226 Calculus Spring 2016 Exam 2V2

DO NOT TURN OVER THIS PAGE UNTIL INSTRUCTED TO DO SO

NAME (Printed): ______________________________

INSTRUCTOR: ______________________________

SECTION NO.: ______________________________

When instructed, turn over this cover page and begin the test. You will have 90 minutes to complete the test. If you have any questions, raise your hand and wait for the proctor to come to your seat.

This test is 6 pages long and contains 7 problems, some with several parts. Write your work on the test papers. If you need extra space, use the backs of the pages and say so on the front. You must show all necessary work for each problem. Solutions presented with no supporting work may receive no credit. Numerical answers should be presented as exact mathematical expressions, simplified as appropriate, not by a decimal approximation, unless explicitly required by the problem.

YOU MAY NOT USE NOTES, CELL PHONES, CALCULATORS OR LAPTOPS AT ANY TIME DURING THE TEST PERIOD. Good luck!

Here are some useful identities.

$$\sin^2(x) = \frac{1-\cos(2x)}{2}, \quad \cos^2(x) = \frac{1+\cos(2x)}{2},$$

$$\int \sec^3(x)dx = \frac{1}{2}[\sec(x)\tan(x) + \ln|\sec(x)+\tan(x)| + C.$$

Problem	Points	Credit
1	35	
2	5	
3	10	
4	10	
5	10	
6	20	
7	10	
Total	100	

(1) (35 Points) Evaluate the following integrals.

(a) (7 Points) $\int \tan^3(x)\ \sec^3(x)\ dx$

(b) (8 Points) $\int \sin^4(3x)\ dx$

(1) (Continued) Evaluate the following integrals.

(c) (10 Points) $\displaystyle\int \frac{2x^3 + 1x^2 + 3x + 6}{x^2(x^2+3)}\, dx$

(d) (8 Points) $\displaystyle\int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}}\, dx$

(2) (5 Points) Circle the correct form for the partial fraction decomposition of the rational function $\dfrac{2x^5+3x^4-x^3+5x^2-x+1}{(x^4-1)(x+1)^2}$:

a) $\dfrac{Ax+B}{x^4-1}+\dfrac{Cx+D}{(x+1)^2}$

b) $\dfrac{A}{x-1}+\dfrac{B}{x+1}+\dfrac{C}{(x+1)^2}+\dfrac{D}{(x+1)^3}+\dfrac{Ex+F}{x^2+1}$

c) $\dfrac{Ax+B}{x^2+1}+\dfrac{C}{(x-1)}+\dfrac{Ex+F}{(x+1)^3}$

d) $\dfrac{Ax+B}{x^2-1}+\dfrac{Cx+D}{x^2+1}+\dfrac{E}{x+1}+\dfrac{F}{(x+1)^2}$

e) $\dfrac{Ax+B}{x^4-1}+\dfrac{C}{x+1}+\dfrac{D}{(x+1)^2}$

f) $\dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}+\dfrac{C}{x+1}+\dfrac{D}{(x+1)^2}+\dfrac{E}{(x+1)^3}$.

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(3) (10 Points) Does $\displaystyle\int_1^\infty \frac{x}{(x^2+1)^5}\,dx$ converge or diverge? Why? Evaluate if it converges.

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(4) (10 Points) Does $\displaystyle\int_{-1}^1 \frac{1}{x^3}\,dx$ converge or diverge? Why? Evaluate if it converges.

(5) (10 Points) Use the Comparison Theorem to determine whether the following improper integral converges or diverges. DO NOT COMPUTE THE EXACT VALUE OF THE INTEGRAL, but show all work needed for the Comparison Theorem.

$$\int_2^{\infty} \frac{x}{\sqrt{x^4 - 1}}\, dx$$

(6) (20 Points) Let the curve C be $y = f(x) = e^{2x}$ for $0 \leq x \leq 1$.

(a) (5 Points) Set up the integral for the arc length of that curve. DO NOT TRY TO EVALUATE OR SIMPLIFY IT.

(b) (15 Points) Find the area of the surface made by revolving that curve about the x-axis. DO EVALUATE THIS INTEGRAL.

(7) (10 points) The curve defined by parametric equations $x = t^2$ and $y = t^3 - 3t$ has been discussed in the textbook. Answer each of the following questions about it. In parts (b), (c) and (d), just set up the appropriate integral but DO NOT EVALUATE OR SIMPLIFY IT. WRITE IT AS AN INTEGRAL INVOLVING EXACTLY ONE VARIABLE, t.

(a) (3 Points) Find the equation of the tangent line to that curve at $t = -2$. SHOW YOUR WORK.

(b) (2 Points) Write the integral for the arclength of the part of the curve where $-\sqrt{3} \leq t \leq \sqrt{3}$,

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(c) (2 Points) Set up the integral for the area between that curve and the x-axis for $0 \leq t \leq \sqrt{3}$.

(d) (3 Points) Set up the integral for the surface area when the curve for $-1 \leq t \leq 1$ is rotated about the y-axis.

Math 226 Calculus Spring 2016 Exam 2V2 Solutions

(1) (35 Points) Evaluate the following integrals.

(a) (7 Points) $\int \tan^3(x)\ \sec^3(x)\ dx = \int \tan^2(x)\ \sec^2(x)\ \tan(x)\sec(x)\ dx$

Let $u = \sec(x)$ so $du = \tan(x)\sec(x)dx$, and $\tan^2(x) = \sec^2(x) - 1 = u^2 - 1$, so the integral becomes

$$\int \tan^3(x)\ \sec^3(x)\ dx = \int \tan^2(x)\ \sec^2(x)\ \tan(x)\sec(x)\ dx = \int (u^2-1)u^2\ du$$

$$= \int (u^4 - u^2)\ du = \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\sec^5(x)}{5} - \frac{\sec^3(x)}{3} + C.$$

(b) (8 Points) $\int \sin^4(3x)\ dx = \int \left(\frac{1-\cos(6x)}{2}\right)^2\ dx$

$$= \frac{1}{4}\int \left(1 - 2\cos(6x) + \cos^2(6x)\right)\ dx = \frac{x}{4} - \frac{\sin(6x)}{12} + \frac{1}{4}\int \left(\frac{1+\cos(12x)}{2}\right) dx$$

$$= \frac{x}{4} - \frac{\sin(6x)}{12} + \frac{1}{8}\left(x + \frac{\sin(12x)}{12}\right) = \frac{3x}{8} - \frac{\sin(6x)}{12} + \frac{\sin(12x)}{96} + C.$$

(c) (10 Points) $\int \frac{2x^3+1x^2+3x+6}{x^2(x^2+3)}\ dx = \int \left(\frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+3}\right)\ dx$ (partial fractions) where

$$2x^3 + 1x^2 + 3x + 6 = (A)x(x^2+3) + B(x^2+3) + (Cx+D)x^2$$

$$= (A+C)x^3 + (B+D)x^2 + (3A)x + 3B$$

gives the equations $A+C=2$, $B+D=1$, $3A=3$, $3B=6$ so $A=1$, $B=2$, $C=1$, $D=-1$. Then the integral is

$$\int \frac{2x^3+1x^2+3x+6}{x^2(x^2+3)}\ dx = \int \frac{1}{x}\ dx + \int \frac{2}{x^2}\ dx + \int \frac{x}{x^2+1}\ dx - \int \frac{1}{x^2+3}\ dx$$

$$= \ln|x| - \frac{2}{x} + \frac{1}{2}\ln|x^2+3| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

$$= \ln\left|\frac{x}{\sqrt{x^2+3}}\right| - \frac{2}{x} - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C \quad \text{either expression is correct.}$$

(d) (8 Points) $\displaystyle\int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}}\, dx$

Since $\cos^2(u) = 1 - \sin^2(u)$, use the trig substitution $x = \sin(u)$ so $\sqrt{1-x^2} = \sqrt{1-\sin^2(u)} = \cos(u)$ and $dx = \cos(u)du$. The limits of integration also change: $x = 0 \leftrightarrow u = 0$ and $x = \frac{1}{2} \leftrightarrow u = \frac{\pi}{6}$ from the usual 30-60-90 degree triangle. Get

$$\int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}}\, dx = \int_0^{\pi/6} \frac{\sin^2(u)\cos(u)\ du}{\cos(u)} = \int_0^{\pi/6} \sin^2(u)\ du = \int_0^{\pi/6} \frac{1-\cos(2u)}{2} du$$

$$= \frac{1}{2}\int_0^{\pi/6} (1-\cos(2u))\ du = \frac{1}{2}\left[u - \frac{\sin(2u)}{2}\right]\Bigg|_0^{\pi/6} = \frac{1}{2}\left[\frac{\pi}{6} - \frac{\sin(\pi/3)}{2}\right] = \frac{\pi}{12} - \frac{\sqrt{3}}{8}.$$

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(2) (5 Points) Since $(x^4-1) = (x^2+1)(x^2-1) = (x^2+1)(x+1)(x-1)$ and each of these factors is irreducible, the denominator factors into irreducibles as $(x-1)(x+1)^3(x^2+1)$, so the correct form for the partial fraction is:

$$\text{b) } \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^3} + \frac{Ex+F}{x^2+1}$$

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(3) (10 Points) Does $\displaystyle\int_1^{\infty} \frac{x}{(x^2+1)^5}\, dx$ converge or diverge? Why? Evaluate if it converges.

This integral is improper since the upper bound is infinity. With substitution $u = x^2+1$, we find the indefinite integral $\displaystyle\int \frac{x}{(x^2+1)^5}\ dx = \frac{1}{2}\int u^{-5}\ du = \frac{-1}{8u^4} + C$ so the improper integral is the converging limit

$$\lim_{t\to\infty} \frac{1}{2}\int_2^t u^{-5}\ du = \lim_{t\to\infty}\left(\frac{-1}{8t^4} - \frac{-1}{8(2^4)}\right) = \lim_{t\to\infty}\left(\frac{1}{128} - \frac{1}{8t^4}\right) = \frac{1}{128}.$$

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(4) (10 Points) Does $\int_{-1}^{1} \frac{1}{x^3}\, dx$ converge or diverge? Why? Evaluate if it converges.

This integral is improper since the denominator of the integrand is zero at $x = 0$. We must break it up into $\int_{-1}^{0} \frac{1}{x^3}\, dx + \int_{0}^{1} \frac{1}{x^3}\, dx$. If either integral diverges, then so does the original. The indefinite integral is $\frac{-1}{2x^2} + C$. Both integrals diverge. The first improper integral diverges since it is defined to be

$$\lim_{t\to 0^-} \int_{-1}^{t} \frac{1}{x^3}\, dx = \lim_{t\to 0^-} \left(\frac{-1}{2t^2} - \frac{-1}{2(-1)^2} \right) = -\infty.$$

The second improper integral diverges since it is defined to be

$$\lim_{s\to 0^+} \int_{s}^{1} \frac{1}{x^3}\, dx = \lim_{s\to 0^+} \left(\frac{-1}{2(1)^2} - \frac{-1}{2(s)^2} \right) = \infty.$$

It is enough to show that one of these improper integrals diverges.

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(5) (10 Points) Use the Comparison Theorem to determine whether the following improper integral converges or diverges. DO NOT COMPUTE THE EXACT VALUE OF THE INTEGRAL, but show all work needed for the Comparison Theorem.

$$\int_{2}^{\infty} \frac{x}{\sqrt{x^4 - 1}}\, dx$$

For $2 \le x$ we have $0 < x^4 - 1 < x^4$ so $0 < \sqrt{x^4 - 1} \le \sqrt{x^4} = x^2$ so $\frac{1}{\sqrt{x^4 - 1}} > \frac{1}{x^2}$ and

$$\frac{x}{\sqrt{x^4 - 1}} > \frac{x}{x^2} = \frac{1}{x} > 0.$$

We know that $\int_{1}^{\infty} \frac{1}{x^p} dx$ diverges for $p \le 1$, so $\int_{1}^{\infty} \frac{1}{x}\, dx$ diverges. By the Comparison Theorem, the given integral diverges.

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(6) (20 Points) Let the curve C be $y = f(x) = e^{2x}$ for $0 \leq x \leq 1$.

(a) (5 Points) Set up the integral for the arc length of that curve. DO NOT TRY TO EVALUATE OR SIMPLIFY IT.

We have $f'(x) = 2e^{2x}$ so $1 + (f'(x))^2 = 1 + 4e^{4x}$ and the integral for the arclength is

$$\int_0^1 \sqrt{1 + (f'(x))^2}dx = \int_0^1 \sqrt{1 + 4e^{4x}}dx$$

(b) (15 Points) Find the area of the surface made by revolving that curve about the x-axis.

That surface area equals

$$\int_0^1 2\pi y\sqrt{1 + (f'(x))^2}dx = \pi \int_0^1 2e^{2x}\sqrt{1 + 4e^{4x}}dx.$$

After the substitution $u = 2e^{2x}$ with $du = 4e^{2x}dx$, the limits of integration are from $u = 2$ to $u = 2e^2$, and the above integral equals $\frac{\pi}{2}\int_2^{2e^2} \sqrt{1 + u^2}du$. Then use the trig sub $u = \tan(\theta)$ so $du = \sec^2(\theta)d\theta$ and the integration limits $2 \leq u \leq 2e^2$ correspond to $\alpha = \tan^{-1}(2) \leq \theta \leq \tan^{-1}(2e^2) = \beta$. This gives (using the integration formula from the cover page of the exam)

$$\frac{\pi}{2}\int_\alpha^\beta \sec^3(\theta)d\theta = \frac{\pi}{2}\frac{1}{2}[\sec(\theta)\tan(\theta) + \ln|\sec(\theta) + \tan(\theta)|]\Big|_\alpha^\beta$$

$$= \frac{\pi}{4}[\sec(\beta)\tan(\beta) + \ln|\sec(\beta) + \tan(\beta)| - \sec(\alpha)\tan(\alpha) - \ln|\sec(\alpha) + \tan(\alpha)|].$$

The triangle corresponding to $2 = \tan(\alpha)$ has side 2 opposite angle α, adjacent side 1 and hypotenuse $\sqrt{5}$, so $\sec(\alpha) = \sqrt{5}$. The triangle corresponding to $2e^2 = \tan(\beta)$ has side $2e^2$ opposite angle β, adjacent side 1 and hypotenuse $\sqrt{1 + 4e^4}$, so $\sec(\beta) = \sqrt{1 + 4e^4}$. The final answer is then

$$\frac{\pi}{4}[2e^2\sqrt{1 + 4e^4} + \ln|2e^2 + \sqrt{1 + 4e^4}| - 2\sqrt{5} - \ln|2 + \sqrt{5}|].$$

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(7) (10 points) The curve defined by parametric equations $x = t^2$ and $y = t^3 - 3t$ has been discussed in the textbook. Answer each of the following questions about it. In parts (b), (c) and (d), just set up the appropriate integral but DO NOT EVALUATE OR SIMPLIFY IT. WRITE IT AS AN INTEGRAL INVOLVING EXACTLY ONE VARIABLE, t.

(a) (3 Points) Find the equation of the tangent line to that curve at $t = -2$. SHOW YOUR WORK.

At $t = -2$ we have the point $(x, y) = (4, -2)$ and the slope of the tangent line is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3(t^2 - 1)}{2t}$$

which equals $\frac{3(3)}{2(-2)} = \frac{9}{-4}$ at $t = -2$ so the equation of the tangent line to the curve at that point is $y + 2 = \frac{-9}{4}(x - 4)$.

(b) (2 Points) Write the integral for the arclength of the part of the curve where $-\sqrt{3} \leq t \leq \sqrt{3}$.

The arclength integral is

$$\int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{(2t)^2 + (3t^2 - 3)^2}\, dt.$$

(c) (2 Points) Set up the integral for the area between that curve and the x-axis for $0 \leq t \leq \sqrt{3}$.

The integral giving that area is $\int_0^3 y dx = \int_0^{\sqrt{3}} (t^3 - 3t)\, 2t dt$.

(d) (3 Points) Set up the integral for the surface area when the curve for $-1 \leq t \leq 1$ is rotated about the y-axis.

The surface area of the curve rotated about the y-axis is

$$\int_{-1}^{1} 2\pi(t^2)\sqrt{(2t)^2 + (3t^2 - 3)^2}\, dt$$
