1. [10]	2. [10]	3. [10]	4. [5]	5. [20]	6. [15]	7[15]	8[15]	Total: [100]

Math 226 **Exam 1** **Feb 15, 2017**

Name: *Section:*

Closed book and closed notes. *Answers must include supporting work.*

Calculators and cell phones out of sight.

1. (10 pts) If $f(x) = 4x - \sin x$ explain why it's one-to-one and find $\left(f^{-1}\right)'(4\pi)$.

2. (10 pts) Find the equation of the tangent line to the curve $y = 5^x$ at the point $(1,5)$.

3. (10 pts) Find the derivative $\frac{dy}{dx}$ for the following:

a) $y = \log_{10}(\cos^2 x)$

b) $y = (\cos x)^x$

4. (5 pts) Write $\tan\left(\arcsin x\right)$ as a ratio of non-trigonometric functions of x.

5. (20 pts) Evaluate the following integrals:

a) $\int_0^{1/2} \frac{\cos^{-1} x}{\sqrt{1-x^2}}\, dx$

b) $\int t^3 \ln t \, dt$

c) $\int \sin^{-1}(3x)\, dx$

6. (15 pts) Evaluate the following limits.

a) $\lim\limits_{t \to 0} \dfrac{e^{2t} - 1}{\sin t}$

b) $\lim\limits_{x \to \infty} \dfrac{\ln x}{\ln(x^2 + 1)}$

c) $\lim\limits_{x \to 0^+} x^x$

7. (15 pts) Find the general antiderivative.

a) $\int \cos^5 x \sin^2 x \, dx$

b) $\int \tan^3\theta \sec^5 \theta \, d\theta$

8. (15 pts) Cobalt-60 has a half-life of 5.24 years.

a) Find the mass that remains from 100-mg sample after 20 years.

b) How long would it take for the mass to decay to 1 mg?

1. [10]	2. [10]	3. [10]	4. [5]	5. [20]	6. [15]	7[15]	8[15]	Total: [100]

Math 226 **Exam 1** **Feb 15, 2017**

Name: Solutions *Section:*

Closed book and closed notes. *Answers must include supporting work.*

Calculators and cell phones out of sight.

1. (10 pts) If $f(x) = 4x - \sin x$ explain why it's one-to-one and find $\left(f^{-1}\right)'(4\pi)$.

$f'(x) = 4 - \cos x > 0$, so $f(x)$ is increasing. Thus, $f(x)$ is 1-1.

$$(f^{-1})'(4\pi) = \frac{1}{f'[f^{-1}(4\pi)]}$$

$4\pi = 4x - \sin x$

$x = \pi$

So, $f^{-1}(4\pi) = \pi$

$$= \frac{1}{f'(\pi)}$$

$$= \frac{1}{4 - \cos\pi}$$

$$= \boxed{\frac{1}{5}}$$

2. (10 pts) Find the equation of the tangent line to the curve $y = 5^x$ at the point $(1,5)$.

$y' = 5^x \ln 5$

$y'(1) = 5\ln 5$

$y_T = (5\ln 5)x + b$

$5 = 5\ln 5 + b$

$b = 5 - 5\ln 5$

$y_T = (5\ln 5)x + (5 - 5\ln 5)$

3. (10 pts) Find the derivative $\frac{dy}{dx}$ for the following:

$$\frac{d}{dx}\log_a x = \frac{1}{x \ln a}$$

a) $y = \log_{10}(\cos^2 x)$

$$y' = \frac{1}{(\cos^2 x)(\ln 10)} \frac{d}{dx}\cos^2 x$$

$$= \frac{-2\cos x \sin x}{\cos^2 x (\ln 10)}$$

$$= -\frac{\sin x}{\cos x}\frac{2}{\ln 10} = \frac{-2\tan x}{\ln 10}$$

b) $y = (\cos x)^x$

$$\ln y = \ln(\cos x)^x$$

$$\ln y = x \ln(\cos x)$$

$$\frac{d}{dx}\ln y = \frac{d}{dx} x \ln(\cos x)$$

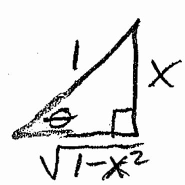
$$\frac{y'}{y} = \ln(\cos x) + x \frac{-\sin x}{\cos x}$$

$$y' = y\left[\ln(\cos x) - x\tan x\right]$$

$$= (\cos x)^x\left[\ln(\cos x - x\tan x\right]$$

4. (5 pts) Write $\tan(\arcsin x)$ as a ratio of non-trigonometric functions of x.

$$\theta = \arcsin x \Leftrightarrow x = \sin\theta$$



$$x^2 + a^2 = 1^2$$

$$a = \sqrt{1-x^2}$$

$$\tan(\arcsin x) = \tan\theta = \boxed{\frac{x}{\sqrt{1-x^2}}}$$

5. (20 pts) Evaluate the following integrals:

a) $\int_0^{1/2} \frac{\cos^{-1} x}{\sqrt{1-x^2}} dx$

Subst.

$$\left[\begin{array}{l} u = \cos^{-1} x \\ du = \frac{-1}{\sqrt{1-x^2}} dx \\ -du = \frac{1}{\sqrt{1-x^2}} dx \end{array}\right]$$

$u(0) = \cos^{-1}(0) = \pi/2$

$u(1/2) = \cos^{-1}(1/2) = \pi/3$

$$= -\int_{\pi/2}^{\pi/3} u\,du = \int_{\pi/3}^{\pi/2} u\,du = \frac{u^2}{2}\Big|_{\pi/3}^{\pi/2} = \boxed{\frac{\pi^2}{8} - \frac{\pi^2}{18}}$$

b) $\int t^3 \ln t\, dt$

By Parts

$\int u\,dv = uv - \int v\,du$

$$\left[\begin{array}{ll} u = \ln t & dv = t^3 dt \\ du = \frac{1}{t} dt & v = \frac{t^4}{4} \end{array}\right]$$

$$= \frac{t^4}{4} \ln t - \frac{1}{4} \int t^3 dt$$

$$= \boxed{\frac{1}{4} t^4 \ln t - \frac{1}{16} t^4 + C}$$

c) $\int \sin^{-1}(3x)\, dx$

By Parts

$$\left[\begin{array}{ll} u = \sin^{-1}(3x) & dv = dx \\ du = \frac{3}{\sqrt{1-9x^2}} dx & v = x \end{array}\right]$$

$$= x \sin^{-1}(3x) - 3 \int \frac{x}{\sqrt{1-9x^2}} dx$$

$$\left[\begin{array}{l} u = 1-9x^2 \\ du = -18x\,dx \\ -\frac{du}{18} = x\,dx \end{array}\right]$$

$$= x \sin^{-1}(3x) + \frac{3}{18} \int u^{-1/2} du$$

$$= \boxed{x \sin^{-1}(3x) + \frac{1}{3}\sqrt{1-9x^2} + C}$$

6. (15 pts) Evaluate the following limits.

a) $\lim_{t\to 0}\frac{e^{2t}-1}{\sin t} \to \left(\frac{0}{0}\right)$

$$\overset{L'H}{=} \lim_{t\to 0}\frac{2e^{2t}}{\cos t} = \frac{2}{1} = \boxed{2}$$

b) $\lim_{x\to\infty}\frac{\ln x}{\ln(x^2+1)} \to \left(\frac{\infty}{\infty}\right)$

$$\overset{L'H}{=} \lim_{x\to\infty}\frac{1/x}{2x/(x^2+1)} = \lim_{x\to\infty}\frac{x^2+1}{2x^2} = \lim_{x\to\infty}\frac{1+1/x^2}{2}$$

$$= \boxed{\frac{1}{2}}$$

c) $\lim_{x\to 0^+} x^x$

$$y = x^x$$
$$\ln y = \ln x^x$$
$$\ln y = x\ln x$$

$$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{1/x} \to \left(\frac{-\infty}{\infty}\right)$$

$$\overset{L'H}{=} \lim_{x\to 0^+}\frac{1/x}{-1/x^2}$$

$$= \lim_{x\to 0^+} -x = 0$$

$$\lim_{x\to 0^+} e^{\ln y} = e^{\lim_{x\to 0^+}\ln y} = e^0 = \boxed{1}$$

7. (15 pts) Find the general antiderivative.

a) $\int \cos^5 x \sin^2 x \, dx$

$= \int \cos^4 x \sin^2 x \cos x \, dx$

$= \int (\cos^2 x)^2 \sin^2 x \cos x \, dx$

$= \int (1-\sin^2 x)^2 \sin^2 x \cos x \, dx$

$\left[\begin{array}{l} u = \sin x \\ du = \cos x \, dx \end{array} \right]$

$= \int (1-u^2)^2 u^2 \, du$

$= \int (1-2u^2+u^4) u^2 \, du$

$= \frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7} + C$

$= \boxed{\frac{\sin^3 x}{3} - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C}$

b) $\int \tan^3\theta \sec^5\theta \, d\theta$

$\tan^2\theta + 1 = \sec^2\theta$

$= \int \tan^2\theta \sec^4\theta \tan\theta \sec\theta \, d\theta$

$= \int (\sec^2\theta - 1) \sec^4\theta \tan\theta \sec\theta \, d\theta$

$\left[\begin{array}{l} u = \sec\theta \\ du = \sec\theta \tan\theta \, d\theta \end{array} \right]$

$= \int (u^2-1) u^4 \, du$

$= \int u^6 - u^4 \, du$

$= \frac{u^7}{7} - \frac{u^5}{5} + C = \boxed{\frac{\sec^7\theta}{7} - \frac{\sec^5\theta}{5} + C}$

8. (15 pts) Cobalt-60 has a half-life of 5.24 years.

a) Find the mass that remains from 100-mg sample after 20 years.

$$M(t) = M_0 e^{kt}$$

$$M(t) = 100 e^{kt}$$

$$M(5.24) = 100 e^{5.24k} = \frac{1}{2}(100) \quad \text{"or"} \quad M_0 e^{5.24k} = \frac{M_0}{2}$$

$$e^{5.24k} = \frac{1}{2} \qquad e^{5.24k} = \frac{1}{2}$$

$$\ln(e^{5.24k}) = \ln(\tfrac{1}{2})$$

$$5.24k = \ln 1 - \ln 2$$

$$\boxed{k = -\frac{\ln 2}{5.24}}$$

$$M(t) = 100 e^{\ln 2 \left(\frac{-t}{5.24}\right)}$$

$$= 100\, 2^{-t/5.24}$$

$$M(20) = \boxed{100\, 2^{-20/5.24}} \text{ mg}$$

b) How long would it take for the mass to decay to 1 mg?

$$100\, 2^{-t/5.24} = 1$$

$$2^{-t/5.24} = 1/100$$

$$\ln 2^{-t/5.24} = \ln(1/100)$$

$$\frac{-t}{5.24} \ln 2 = \ln 1 - \ln 100$$

$$\frac{-t}{5.24} = \frac{-\ln 100}{\ln 2}$$

$$\boxed{t = \left(\frac{\ln 100}{\ln 2}\right) 5.24} \text{ years}$$