Math 225 **(White)** **Midterm Exam**
Differential Calculus **February 19, 2016**

Do NOT turn over this page until instructed to begin.

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Section: ∞

Binghamton ID Number: B52

Instructions

Write clear, careful, neat solutions to the questions in the space provided.

No books, no notes, no electronic devices (calculators, cell phones, smart watches, etc.) allowed!

Write all your work on the test–nothing else will be graded. **You must show all your work and justify your answers.** Your work must be legible, and the final answers must be reasonably simplified.

On some problems you may be asked to use a specific method to solve the problem (for instance, "Use the Fundamental Theorem of Calculus to find..."). On all other problems, you may use any method we have covered. You may not use methods that we have not covered.

Wandering Eyes Policy

You must keep your eyes on your own work at all times. If you are found looking around, you will be warned once, and only once. A second infraction may result in automatic zero on this test, and possibly a referral to the Harpur College Academic Honesty Committee.

Duration of the Test

This is a timed test designed for one class period. You will start the test when your proctor tells you to start, and you will finish the test when your proctor tells you to stop, when the class period is over.

1	2	3	4	5	6	7	8	9	Total
15 pts	15 pts	8 pts	5 pts	5 pts	10 pts	15 pts	12 pts	15 pts	100 pts

1. (15 points) Find the most general antiderivatives.

(a) $\int (3x-5)(x^2+1)\ dx$

$= \int 3x^3 - 5x^2 + 3x - 5\ dx$

$= 3\cdot\frac{1}{4}x^4 - 5\cdot\frac{1}{3}x^3 + 3\cdot\frac{1}{2}x^2 - 5x + C$

$= \frac{3}{4}x^4 - \frac{5}{3}x^3 + \frac{3}{2}x^2 - 5x + C$

(b) $\int \frac{\sec^2(x)}{\sqrt{5\tan(x)+3}}\ dx$

$u(x) = 5\tan(x) + 3$

$u'(x) = 5\sec^2(x)$

$= \frac{1}{5}\int \frac{5\sec^2(x)}{\sqrt{5\tan(x)+3}}\ dx$

$= \frac{1}{5}\int \frac{u'(x)}{\sqrt{u(x)}}\ dx$

$= \frac{1}{5}\int \frac{1}{\sqrt{u}}\ du$

$= \frac{1}{5}\int u^{-\frac{1}{2}}\ du$

$= \frac{1}{5}\left[2(u(x))^{\frac{1}{2}}\right] + C$

$= \frac{2}{5}\sqrt{5\tan(x)+3} + C$

(c) $\int x^3\sqrt{x^2+9}\ dx$

$u(x) = x^2 + 9$

$u'(x) = 2x$

$x^2 = u(x) - 9$

$= \frac{1}{2}\int \sqrt{x^2+9}\cdot x^2\cdot 2x\ dx$

$= \frac{1}{2}\int \sqrt{u(x)}\ (u(x)-9)\ u'(x)\ dx$

$= \frac{1}{2}\int \sqrt{u}\ (u-9)\ du$

$= \frac{1}{2}\int u^{3/2} - 9u^{1/2}\ du$

$= \frac{1}{2}\cdot\frac{2}{5}(u(x))^{5/2} - \frac{9}{2}\cdot\frac{2}{3}(u(x))^{\frac{3}{2}} + C$

$= \frac{1}{5}(x^2+9)^{5/2} - 3(x^2+9)^{3/2} + C$

2. (15 points) Evaluate and simplify.

(a) $\int_0^2 x^2 - 2\sqrt{x}\ dx$

$= \int_0^2 x^2 - 2x^{\frac{1}{2}}\, dx$

$= \frac{1}{3}x^3 - 2\cdot\frac{2}{3}x^{3/2} \Big]_0^2$

$= \left[\frac{1}{3}(2^3) - \frac{4}{3}(2^{3/2})\right] - \left[\frac{1}{3}\cdot 0^3 - \frac{4}{3}\cdot 0^3\right]$

$= \frac{8}{3} - \frac{4\sqrt{8}}{3} \quad = \quad \frac{8 - 8\sqrt{2}}{3}$

(b) $\int_0^{\pi/2} \sin(x)\cos(x)\ dx$

$u(x) = \sin(x)$

$u'(x) = \cos(x)$

$u(0) = \sin(0) = 0$

$u(\pi/2) = \sin(\pi/2) = 1$

$= \int_0^{\pi/2} u(x)\cdot u'(x)\, dx$

$= \int_0^1 u\ du$

$= \frac{u^2}{2}\Big]_0^1 \quad = \quad \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2}$

(c) $\int_0^{\sqrt{4\pi}} x\sin(x^2)\ dx$

$u(x) = x^2$

$u'(x) = 2x$

$u(0) = 0$

$u(\sqrt{4\pi}) = 4\pi$

$= \frac{1}{2}\int_0^{\sqrt{4\pi}} \sin(u(x))\, u'(x)\, dx$

$= \frac{1}{2}\int_0^{4\pi} \sin(u)\, du$

$= -\frac{1}{2}\cos(u)\Big]_0^{4\pi}$

$= \left(-\frac{1}{2}\cos(4\pi)\right) - \left(-\frac{1}{2}\cos(0)\right) \quad = \quad -\frac{1}{2} - \left(-\frac{1}{2}\right) = 0$

3. (8 points) For each function $g(x)$ given below, compute the derivative $g'(x)$.

(a) $g(x) = \int_{x^2+1}^{10} \frac{\sin(t^2)}{t-1}\, dt \quad = \quad -\int_{10}^{x^2+1} \frac{\sin(t^2)}{t-1}\, dt$

$g'(x) = \ldots$

$$g'(x) = -\left[\frac{\sin((x^2+1)^2)}{(x^2+1)-1}\right] \cdot 2x$$

$$= \frac{-2x\sin((x^2+1)^2)}{x^2}$$

$$= -\frac{2\sin((x^2+1)^2)}{x}$$ by the FTC and Chain Rule.

(b) $g(x) = x + \int_{2}^{10} t^2 + 3t\, dt$

$g'(x) = \ldots$

$g'(x) = 1$ because $\frac{d}{dx}\int_2^{10} t^2+3t\,dt = 0$

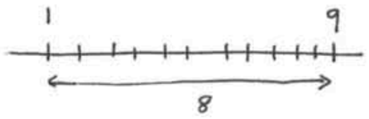
since $\int_2^{10} t^2+3t\,dt$ is a constant,

so its derivative is 0.

4. (5 points) Express the following limit as an integral:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left[\sqrt{\left(1+\frac{8i}{n}\right)^2 + 3\left(1+\frac{8i}{n}\right)} \right] \left(\frac{8}{n}\right)$$

$$\int_1^9 \sqrt{x^2 + 3x}\, dx$$



5. (5 points) An oil tanker has developed a leak. Oil is leaking from the tanker at a rate of $r(t) = 800 - 6t^2$ liters per minute. How much oil has leaked out of the tanker after 10 minutes?

$$\int_0^{10} 800 - 6t^2\, dt = 800t - 6 \cdot \tfrac{1}{3} t^3 \Big]_0^{10}$$

$$= 800t - 2t^3 \Big]_0^{10}$$

Net change theorem.

$$= [8000 - 2(1000)] - 0$$

$$= 8000 - 2000$$

$$= 6000 \text{ liters}$$

6. (10 points) A particle is moving along the x-axis, with its initial position at the origin and initial velocity -12 meters per second. It is known that the acceleration of the particle, at all times t *after* the initial time, is $a(t) = 6t - 9$.

(a) Find the position $s(t)$ of the particle at arbitrary time t.

$$a(t) = v'(t) = 6t - 9$$

$$v(t) = 6 \cdot \tfrac{1}{2} t^2 - 9t + c = 3t^2 - 9t + c$$

$$-12 = v(0) = 0 + c = c$$

$$v(t) = 3t^2 - 9t - 12$$

$$v(t) = s'(t) = 3t^2 - 9t - 12$$

$$s(t) = 3 \cdot \tfrac{1}{3} t^3 - 9 \cdot \tfrac{1}{2} t^2 - 12t + d$$

$$= t^3 - \tfrac{9}{2} t^2 - 12t + d$$

$$0 = s(0) = 0 + d = d$$

$$s(t) = t^3 - \tfrac{9}{2} t^2 - 12t$$

(b) Are there any times at which the particle is not moving? If so, find them. If not, explain briefly.

$v(t) = 0$ when $3t^2 - 9t - 12 = 0$

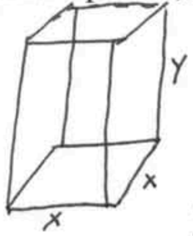
$$3t^2 - 9t - 12 = 3(t^2 - 3t - 4)$$

$$= 3(t-4)(t+1) = 0$$

when $t = 4$ and when $t = -1$.

However, motion is unknown when $t < 0$, so we only know the particle is not moving when $t = 4$.

7. (15 points) A box with square base and open top is to be constructed. Material for the sides costs 2 dollars per square foot and material for the reinforced bottom costs 5 dollars per square foot. What are the dimensions of the largest volume box you can build for 375 dollars? [Be sure to draw a picture, define your symbols, and show your work.]



x = length & width of square base

y = height $\quad C = \text{cost} = 375 \quad V = \text{volume}$

$$375 = C = 2 \cdot 4xy + 5x^2 = 8xy + 5x^2$$

$$V = x^2 y$$

$$\boxed{\begin{array}{l}\text{Final answer:} \\ x = 5 \text{ feet} \\ y = \frac{25}{4} \text{ feet}\end{array}}$$

$$8xy = 375 - 5x^2$$

$$y = \frac{375 - 5x^2}{8x}$$

$$V(x) = x^2\left(\frac{375-5x^2}{8x}\right)$$

$$= \frac{375}{8}x - \frac{5}{8}x^3$$

$$\text{dom}(V) = (0, 5\sqrt{3})$$

Note: when $y = 0$,

$$5x^2 = 375$$
$$x^2 = 75$$
$$x = \sqrt{75} = 5\sqrt{3}$$

(x must be positive)

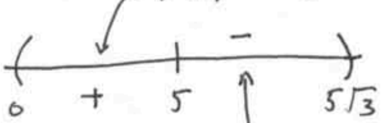
Critical points occur when $V'(x) = 0$.

$$V'(x) = \frac{375}{8} - \frac{15}{8}x^2$$

$$= \frac{15}{8}(25 - x^2) = \frac{15}{8}(5-x)(5+x)$$

$V'(x) = 0$ when $x = 5$ and when $x = -5$, but $-5 \notin \text{dom}(V)$.

$V'(1) = \frac{360}{8} > 0$



$V'(6) = \frac{15}{8}(-1)(11) = \frac{-165}{8} < 0$

Perhaps easier to use 2nd derivative test: $V''(x) = \frac{-30}{8}x$ always negative.

By first derivative test maximum at $x = 5$, and $y = \frac{375 - 5(25)}{40} = \frac{250}{40} = \frac{25}{4}$.

This is absolute maximum.

8. (12 points) Let $f(x)$ be a function such that $\int_2^8 g(x)\,dx = 5$ and $\int_2^4 g(x)\,dx = -6$.
Below is a list of integrals. For each part:

- If you have enough information to evaluate the integral, then do so, showing your work.
- If you dont have enough information to evaluate the integral, write "can't be done". You do not need to explain why it can't be done

(a) $\int_8^4 g(x)\,dx$

$$= \int_8^2 g(x)\,dx + \int_2^4 g(x)\,dx$$

$$= -\int_2^8 g(x)\,dx + \int_2^4 g(x)\,dx$$

$$= -5 + (-6) = -11.$$

(b) $\int_1^2 g(2x)\,dx$

$$= \frac{1}{2}\int_1^2 g(u(x))\,u'(x)\,dx$$

$u(x) = 2x$

$u'(x) = 2$

$u(1) = 2$

$u(2) = 4$

$$= \frac{1}{2}\int_2^4 g(u)\,du = -3$$

(c) $\int_2^8 3g(x) + 4\,dx$

$$= 3\int_2^8 g(x)\,dx + \int_2^8 4\,dx$$

$$= 3(5) + 4x\Big]_2^8$$

$$= 15 + (32-8) = 39$$

(d) $\int_2^8 (g(x))^2\,dx$

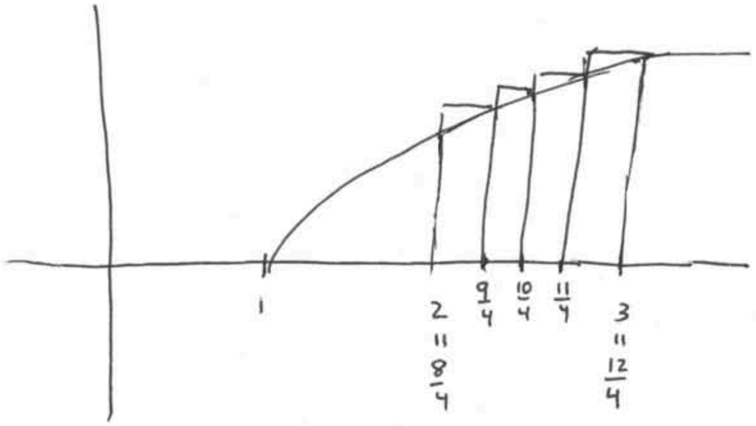
Can't be done.

9. (15 points) All three parts of the problem concern the integral

$$\int_2^3 \sqrt{x-1}\ dx.$$

(a) Draw the graph of $f(x) = \sqrt{x-1}$ along with the rectangles required to compute the Riemann sum R_4 using right-hand points.

$\frac{3-2}{4} = \frac{1}{4} = \Delta x$



(b) Write down the Riemann sum R_4 explicitly as a sum of real numbers. **Do not simplify.**

$$\frac{1}{4}\left(\sqrt{\frac{9}{4}-1}\right) + \frac{1}{4}\left(\sqrt{\frac{10}{4}-1}\right) + \frac{1}{4}\left(\sqrt{\frac{11}{4}-1}\right) + \frac{1}{4}\left(\sqrt{\frac{12}{4}-1}\right)$$

$$= \frac{1}{4}\sqrt{\frac{5}{4}} + \frac{1}{4}\sqrt{\frac{6}{4}} + \frac{1}{4}\sqrt{\frac{7}{4}} + \frac{1}{4}\sqrt{\frac{8}{4}}$$

$$= \frac{1}{8}\left(\sqrt{5}+\sqrt{6}+\sqrt{7}+\sqrt{8}\right)$$

(c) Is the Riemann sum R_4 *greater than*, *less than*, or *equal to* the definite integral? Explain briefly.

R_4 is greater than $\int_2^3 \sqrt{x-1}\,dx$ because $f(x) = \sqrt{x-1}$ is increasing, so the right hand points in each interval are maxima.