Sols Version 1 (8:00 AM)

1. (15 points) Find the limit. If the limit does not exist because it is infinite, indicate positive or negative infinity.

a)

$$\lim_{x\to -2^+} \frac{2x+1}{x^2+2x}$$

$$= \lim_{x\to -2^+} \frac{2x+1}{x(x+2)}$$

$$= \infty$$

b)

$$\lim_{x\to -2^+} \frac{3-\sqrt{7-x}}{x^2+2x} \quad \frac{3+\sqrt{7-x}}{3+\sqrt{7-x}}$$

$$= \lim_{x\to -2^+} \frac{9-(7-x)}{(x^2+2x)(3+\sqrt{7-x})}$$

$$= \lim_{x\to -2^+} \frac{\cancel{2+x}}{x(\cancel{x+2})(3+\sqrt{7-x})}$$

$$= \lim_{x\to -2^+} \frac{1}{x(3+\sqrt{7-x})}$$

$$= \frac{-1}{12}$$

2. (15 points) Find the limit. If the limit does not exist, because it is infinite, indicate positive or negative infinity.

a)

$$\lim_{x\to 0} \frac{\sin(7x)}{x^2 - 3x}$$

$$= \lim_{x\to 0} \frac{\sin(7x)}{x(x-3)}$$

$$= 7 \lim_{x\to 0} \frac{\sin(7x)}{7x} \cdot \frac{1}{x-3}$$

$$= 7(1)\left(\frac{-1}{3}\right)$$

$$= -7/3$$

b)

$$\lim_{x\to 3^-} \frac{x^2 - 4x + 3}{|x-3|}$$

$$= \lim_{x\to 3^-} \frac{\cancel{(x-3)}(x-1)}{-\cancel{(x-3)}}$$

$$= \lim_{x\to 3^-} -(x-1)$$

$$= -2$$

3. (10 points) Find the equation of the tangent line to the graph of g at the point $(2, g(2))$.

$$g(x) = x \cdot \sqrt{3x-2} = x(3x-2)^{1/2}$$

You may leave your answer in point-slope form.

$y_T = mx + b$

$y_T = \frac{7}{2}x + b$

plug in $(2, 4)$:

$4 = \frac{7}{2}(2) + b$

$b = -3 \Rightarrow \boxed{y_T = \frac{7}{2}x - 3}$

$g'(x) = (1)(3x-2)^{1/2} + \frac{1}{2}x(3x-2)^{-1/2}(3)$

$= \sqrt{3x-2} + \frac{3x}{2\sqrt{3x-2}}$

$m = g'(2) = 2 + \frac{6}{2(2)}$

$= 7/2$

4. (10 points) An object moves along a line so that its position at time t (seconds) is given by $s(t) = 3t^2 + t$ (meters).

(a) Find the velocity and acceleration of the object at time t.

$v(t) = s'(t) = 6t + 1$ m/s

$a(t) = s''(t) = 6$ m/s²

(b) Find the position of the object when the velocity is 13 m/s.

$v(t) = 6t + 1 = 13$

$6t = 12$

$t = 2$ secs.

$s(2) = 3(2)^2 + 2$

$= 14$ meters

5. (15 points) Find the derivatives (by any method). You do not need to simplify your answer.

a) $f(x) = \dfrac{2x-1}{\sin^2(x)}$. $\quad f'(x) = \dfrac{2\sin^2 x - (2x-1)(2)\sin x \cos x}{\sin^4 x}$

b) $f(x) = x^4 \cdot \sqrt{x^5 + 3\tan(x)}$. $\quad f'(x) = 4x^3\sqrt{x^5+3\tan x} + x^4\left(\frac{1}{2}\right)(x^5+3\tan x)^{-1/2} \cdot \frac{d}{dx}(x^5+3\tan x)$

$$f'(x) = 4x^3\sqrt{x^5+3\tan x} + \frac{x^4(5x^4+3\sec^2 x)}{2\sqrt{x^5+3\tan x}}$$

c) $f(x) = \cos\left(\dfrac{2x-1}{x^3+x}\right)$. $\quad f'(x) = -\sin\left(\frac{2x-1}{x^3+x}\right)\frac{d}{dx}\left(\frac{2x-1}{x^3+x}\right)$

$$= -\sin\left(\frac{2x-1}{x^3+x}\right) \cdot \left[\frac{2(x^3+x) - (3x^2+1)(2x-1)}{(x^3+x)^2}\right]$$

6. (20 points) Function $f(x)$ is defined below.

$$f(x) = \begin{cases} x^2+1, & \text{if } x < 0 \\ \dfrac{\sin x}{x}, & \text{if } 0 < x < 2 \\ 0, & \text{if } 2 \le x \end{cases}$$

a) Find, with explanation, **all** real numbers a, such that f(x) is not continuous at a.

- f(x) is not cont. @ x=0 due to f(0) not being defined.

- f(x) is not cont. @ x=2 due to

$$\lim_{x\to 2^-} f(x) = \frac{\sin 2}{2} \neq \lim_{x\to 2^+} f(x) = 0$$

b) For each such a, find $\lim_{x\to a} f(x)$ or explain why it does not exist.

$$\left.\begin{aligned} \lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-} x^2+1 = 1 \\ \lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} \frac{\sin x}{x} = 1 \end{aligned}\right\} \Rightarrow \lim_{x\to 0} f(x) = 1$$

$$\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x) \quad \text{so} \quad \lim_{x\to 2} f(x) \text{ DNE}$$

7. (15 points) Find the derivative of the function

$$f(x) = \frac{1}{4x-1}$$

using the limit definition of the derivative.

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{4(x+h)-1} - \frac{1}{4x-1}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{4(x+h)-1} - \frac{1}{4x-1}}{h} \cdot \frac{[4(x+h)-1][4x-1]}{[4(x+h)-1][4x-1]}$$

$$= \lim_{h \to 0} \frac{4x-1-[4(x+h)-1]}{h[4(x+h)-1][4x-1]}$$

$$= \lim_{h \to 0} \frac{-4\cancel{h}}{\cancel{h}[4(x+h)-1][4x-1]}$$

$$= \lim_{h \to 0} \frac{-4}{[4(x+h)-1][4x-1]}$$

$$= \frac{-4}{(4x-1)^2}$$

Version 2 (11:20)

1. (15 points) Find the limit. If the limit does not exist, because it is infinite, indicate positive or negative infinity.

a)

$$\lim_{x\to 0} \frac{\sin(5x)}{x^2-8x}$$

$$= \lim_{x\to 0} \frac{\sin(5x)}{x(x-8)}$$

$$= 5\lim_{x\to 0} \frac{\sin 5x}{5x} \cdot \frac{1}{x-8}$$

$$= 5(1)\left(\frac{-1}{8}\right)$$

$$= -5/8$$

b)

$$\lim_{x\to 2^-} \frac{x^2+x-6}{|x-2|}$$

$$= \lim_{x\to 2^-} \frac{(x+3)(x-2)}{|x-2|}$$

$$= \lim_{x\to 2^-} \frac{(x+3)(x-2)}{-(x-2)}$$

$$= \lim_{x\to 2^-} -(x+3)$$

$$= -5$$

2. (15 points) Find the limit. If the limit does not exist because it is infinite, indicate positive or negative infinity.

a)

$$\lim_{x \to -3^-} \frac{3 - \sqrt{6 - x}}{x^2 + 3x} \cdot \frac{3+\sqrt{6-x}}{3+\sqrt{6-x}}$$

$$= \lim_{x \to -3^-} \frac{9-(6-x)}{x(x+3)(3+\sqrt{6-x})}$$

$$= \lim_{x \to -3^-} \frac{\cancel{3+x}}{x\cancel{(x+3)}(3+\sqrt{6-x})}$$

$$= \lim_{x \to -3^-} \frac{1}{x(3+\sqrt{6-x})} = \boxed{-\frac{1}{18}}$$

b)

$$\lim_{x \to -3^-} \frac{3x + 1}{x^2 + 3x}$$

$$= \lim_{x \to -3^-} \frac{3x+1}{x(x+3)}$$

$$= \infty$$

3. (10 points) An object moves along a line so that its position at time t (seconds) is given by $s(t) = 5t^2 - 3t$ (meters).

(a) Find the velocity and acceleration of the object at time t.

$$v(t) = s'(t) = 10t - 3 \quad m/s$$

$$a(t) = s''(t) = 10 \quad m/s^2$$

(b) Find the position of the object when the velocity is 27 m/s.

$$v(t) = 10t - 3 = 27$$

$$10t = 30$$

$$t = 3 \text{ secs.}$$

$$s(3) = 5(3)^2 - 3(3) = 36 \text{ meters}$$

4. (10 points) Find the equation of the tangent line to the graph of h at the point $(1, h(1))$.

$$h(x) = x^2 \cdot \sqrt{3x+1}$$

You may leave your answer in point-slope form.

$$h'(x) = 2x\sqrt{3x+1} + x^2 \frac{d}{dx}(3x+1)^{1/2}$$

$$= 2x\sqrt{3x+1} + \frac{x^2}{2}(3x+1)^{-1/2}(3)$$

$$= 2x\sqrt{3x+1} + \frac{3x^2}{2\sqrt{3x+1}}$$

$$m = h'(1) = 2(2) + \frac{3}{2(2)}$$

$$= 19/4$$

$$y_T = mx + b$$

$$y_T = \frac{19}{4}x + b$$

Plug in $(1, 2)$:

$$2 = \frac{19}{4}(1) + b$$

$$b = \frac{-15}{4}$$

$$\boxed{y_T = \frac{19}{4}x - \frac{15}{4}}$$

5. (20 points) Function $f(x)$ is defined below.

$$f(x) = \begin{cases} x^2+1, & \text{if } x \le 0 \\ \dfrac{\sin x}{x}, & \text{if } 0 < x < 2 \\ 0, & \text{if } 2 \le x \end{cases}$$

a) Find, with explanation, **all** real numbers a, such that f(x) is not continuous at a.

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^2+1 = 1$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{\sin x}{x} = 1$

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{\sin x}{x} = \frac{\sin 2}{2}$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{\sin x}{x} = 0$

Not Cont. @ $x=2$

Since $\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$

b) For each such a, find $\lim_{x \to a} f(x)$ or explain why it does not exist.

$\lim_{x \to 0} f(x) = 1$

$\lim_{x \to 2} f(x)$ DNE due to

$\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$

6. (15 points) Find the derivatives (by any method). You do not need to simplify your answer.

a) $f(x) = x^5 \cdot \sqrt{x^4 + 3\cos(x)}$. $\quad f'(x) =$ $5x^4\sqrt{x^4+3\cos x} + x^5 \frac{d}{dx}(x^4+3\cos x)^{1/2}$

$f'(x) = 5x^4\sqrt{x^4+3\cos x} + x^5\left(\frac{1}{2}\right)(x^4+3\cos x)^{-1/2}(4x^3-3\sin x)$

b) $f(x) = \dfrac{3x+5}{\sin^2(x)}$. $\quad f'(x) =$ $\dfrac{3(\sin^2 x) - (3x+5)\frac{d}{dx}(\sin^2 x)}{\sin^4 x}$

$= \dfrac{3\sin^2 x - (3x+5)(2)\sin x \cos x}{\sin^4 x}$

c) $f(x) = \tan\left(\dfrac{2x-1}{x^3+x}\right)$. $\quad f'(x) =$ $\sec^2\left(\frac{2x-1}{x^3+x}\right) \cdot \frac{d}{dx}\left(\frac{2x-1}{x^3+x}\right)$

$f'(x) = \sec^2\left(\frac{2x-1}{x^3+x}\right) \cdot \left[\dfrac{2(x^3+x) - (2x-1)(3x^2+1)}{(x^3+x)^2}\right]$

7. (15 points) Find the derivative of the function

$$f(x) = \frac{1}{3x+2}$$

using the limit definition of the derivative.

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{3(x+h)+2} - \frac{1}{3x+2}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{3(x+h)+2} - \frac{1}{3x+2}}{h} \cdot \frac{[3(x+h)+2](3x+2)}{[3(x+h)+2](3x+2)}$$

$$= \lim_{h \to 0} \frac{3x+2-[3(x+h)+2]}{h[3(x+h)+2][3x+2]}$$

$$= \lim_{h \to 0} \frac{-3\cancel{h}}{\cancel{h}[3(x+h)+2][3x+2]}$$

$$= \lim_{h \to 0} \frac{-3}{[3(x+h)+2][3x+2]}$$

$$= \frac{-3}{(3x+2)^2}$$