

Math 224

Midterm

Nov 20, 2015

Name:

Section:

Closed book and closed notes.

Answers must include supporting work.

Calculators and cell phones out of sight.

1. (10 pts) Using the definition of derivative, find $f'(x)$ if $f(x) = \frac{1}{x}$. Do not use differentiation formulas. Assume $x \neq 0$.

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\left(\frac{1}{x+h} - \frac{1}{x}\right)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{1}{x+h} - \frac{1}{x}\right) \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}\right) \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{x-x-h}{x(x+h)}\right) \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{-h}{x(x+h)}\right) \\
 &= \lim_{h \rightarrow 0} \frac{-h}{h} \cdot \lim_{h \rightarrow 0} \frac{1}{x(x+h)} \quad (\text{because both limits exist}) \\
 &= \lim_{h \rightarrow 0} (-1) \cdot \lim_{h \rightarrow 0} \frac{1}{x(x+h)} \quad (\text{because limits do not depend on the value at the limit point: limits are local}) \\
 &= (-1) \cdot \frac{1}{x(x+0)} \quad (\text{continuity of rational functions}) \\
 &= \frac{-1}{x^2}
 \end{aligned}$$

2. (15 pts) In each of the following calculate $\frac{dy}{dx}$.

a) $y = \sin \sqrt{2x+1}$

$$f(x) = \sin(\sqrt{2x+1}) \quad \text{Chain rule}$$

$$\begin{aligned} f'(x) &= \cos(\sqrt{2x+1}) \cdot \frac{d}{dx} \sqrt{2x+1} \\ &= \cos(\sqrt{2x+1}) \cdot \frac{1}{2\sqrt{2x+1}} \cdot \frac{d}{dx}(2x+1) \\ &= \cos(\sqrt{2x+1}) \cdot \frac{1}{2\sqrt{2x+1}} \cdot 2 \\ &= \frac{\cos(\sqrt{2x+1})}{\sqrt{2x+1}} \end{aligned}$$

b) $y = \frac{ax+b}{cx+d}$

Quotient Theorem

$$f(x) = \frac{ax+b}{cx+d}$$

$$\begin{aligned} f'(x) &= \frac{(cx+d) \cdot \frac{d}{dx}(ax+b) - (ax+b) \frac{d}{dx}(cx+d)}{(cx+d)^2} \\ &= \frac{a(cx+d) - c(ax+b)}{(cx+d)^2} \\ &= \frac{ad - bc}{(cx+d)^2} \end{aligned}$$

c) $y = \tan(x^2 \sin x)$

$$f(x) = \tan(x^2 \sin x)$$

$$\begin{aligned} f'(x) &= \sec^2(x^2 \sin x) \cdot \frac{d}{dx} x^2 \sin x \\ &= \sec^2(x^2 \sin x) \cdot \left[x^2 \cdot \frac{d}{dx} \sin x + \sin x \frac{d}{dx} x^2 \right] \\ &= \sec^2(x^2 \sin x) \cdot \left[x^2 (\cos x) + \sin x (2x) \right] \\ &= \sec^2(x^2 \sin x) \cdot [2x \sin x + x^2 \cos x] \end{aligned}$$

3. (15 pts) The position of a particle on a straight line is given by $s(t) = t^3 - 3t$, where s is in meters and t is in seconds. Find

- the velocity and acceleration as functions of t
- the acceleration after 2 seconds
- the acceleration when the velocity is 0

(a) $v(t) = s'(t) = 3t^2 - 3 \quad \leftarrow \text{velocity}$
 $a(t) = s''(t) = 6t \quad \leftarrow \text{acceleration}$

(b) $a(2) = 12 \text{ m/sec}^2$

(c) $v(t) = 0 \text{ when } 3t^2 - 3 = 0$
 $3(t^2 - 1) = 0$
 $3(t-1)(t+1) = 0$

Thus, velocity is zero when $t=1$ and when $t=-1$.

Hence, $a(1) = 6 \cdot 1 = 6$
 $a(-1) = 6 \cdot (-1) = -6.$

4. (10 pts)

a) For what value of the constant c is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geq 2 \end{cases}$$

Since polynomials are continuous, $f(x)$ is continuous for all $x < 2$ and all $x > 2$ regardless of the value of c . The only remaining point to test is $x = 2$.

For continuity at c , need $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2} f(x) = f(2)$.
 Now, $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} cx^2 + 2x = c \cdot 2^2 + 2 \cdot 2 = 4c + 4$.

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} x^3 - cx = 2^3 - c \cdot 2 = 8 - 2c.$$

$$f(2) = 2^3 - 2c = 8 - 2c.$$

Thus $f(x)$ is continuous at 2 (and hence on $(-\infty, \infty)$) when $4c + 4 = 8 - 2c \quad \underbrace{\quad}_{6c = 4} \quad c = \frac{4}{6} = \frac{2}{3}$.

5. (15 pts) Evaluate the following limits:

a) $\lim_{\theta \rightarrow 0} \frac{\sin 5\theta}{3\theta}$

b) $\lim_{x \rightarrow 2^-} \frac{x^2 - 4}{|x - 2|}$

c) $\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x}$

[Hint: use Squeeze Thm]

(a) $\lim_{\theta \rightarrow 0} \frac{\sin(5\theta)}{3\theta} = \lim_{\theta \rightarrow 0} \frac{5}{5} \cdot \frac{\sin(5\theta)}{5\theta} = \frac{5}{3} \cdot \lim_{\theta \rightarrow 0} \frac{\sin(5\theta)}{5\theta} = \frac{5}{3} \cdot 1 = \frac{5}{3}$

(b) $\lim_{x \rightarrow 2^-} \frac{x^2 - 4}{|x - 2|} = \lim_{x \rightarrow 2^-} \frac{(x-2)(x+2)}{-(x-2)} = \lim_{x \rightarrow 2^-} -(x+2) = -4$

(c) Since $-1 \leq \sin(\frac{1}{x}) \leq 1$, we see $-x^2 \leq x^2 \sin(\frac{1}{x}) \leq x^2$.

Also $\lim_{x \rightarrow 0} x^2 = 0 = \lim_{x \rightarrow 0} -(x^2)$. By the Squeeze Theorem,

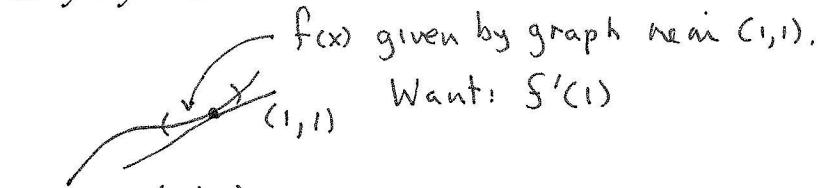
$$\lim_{x \rightarrow 0} x^2 \sin(\frac{1}{x}) = 0.$$

6. (15 pts) Find the equation of the tangent line to the curve defined by

$$x^2 + 3x^2y + y^2 = 5$$

at the point $(x, y) = (1, 1)$.

$$x^2 + 3x^2 f(x) + [f(x)]^2 = 5$$



Differentiate: $2x + 3x^2 \cdot f'(x) + f(x) \cdot 6x + 2f(x)f'(x) = 0$

Specialize: $2 \cdot 1 + 3 \cdot 1^2 f'(1) + 1 \cdot 6 \cdot 1 + 2 \cdot 1 \cdot f'(1) = 0$

to $(1, 1)$

$$2 + 3 \cdot f'(1) + 6 + 2 \cdot f'(1) = 0$$

$f(1) = 1$.

$$5 \cdot f'(1) + 8 = 0$$

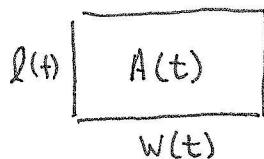
$$5f'(1) = -8$$

$$f'(1) = -\frac{8}{5}$$

Tangent line: $y - 1 = \left(-\frac{8}{5}\right)(x - 1)$

$$y = \left(-\frac{8}{5}\right)x + \frac{8}{5} + 1 \quad y = \left(-\frac{8}{5}\right)x + \frac{13}{5}$$

7. (10 pts) The length of a rectangle is increasing at a rate of 8 cm/s and its width is increasing at a rate of 3 cm/s. When the length is 20 cm and the width is 10 cm, how fast is the area of the rectangle increasing?



Let t_0 be the time at which
 $l(t_0) = 20$ and $w(t_0) = 10$.

Know: $l'(t_0) = 8$ Want: $A'(t_0)$

$$w'(t_0) = 3$$

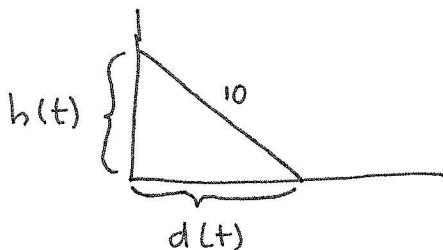
Relation: $A(t) = l(t) \cdot w(t)$

Differentiate: $A'(t) = l(t)w'(t) + w(t)l'(t)$

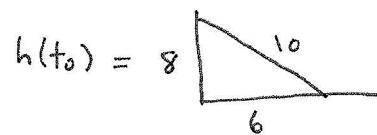
Specialize: $A'(t_0) = l(t_0)w'(t_0) + w(t_0)l'(t_0)$

$$\begin{aligned} &= 20 \cdot 3 + 10 \cdot 8 \\ &= 60 + 80 \\ &= \underline{\underline{140 \text{ cm}^2/\text{sec.}}} \end{aligned}$$

8. (10 pts) A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?



Let t_0 be the time at which
 $d(t_0) = 6$. At time t_0 snapshot:



Know: $d(t_0) = 6$ Want: $h'(t_0)$ $100 - 36 = 64$
 $d'(t_0) = 1$

Relation: $d^2(t) + h^2(t) = 100$

Differentiate: $2d(t)d'(t) + 2h(t)h'(t) = 0$

Specialize: $2d(t_0)d'(t_0) + 2h(t_0)h'(t_0) = 0$

$$2 \cdot 6 \cdot 1 + 2 \cdot 8 \cdot h'(t_0) = 0$$

$$16h'(t_0) = -12$$

$$h'(t_0) = -\frac{12}{16}$$

$$= -\frac{3}{4} \text{ ft/sec.}$$