Math 223                                Midterm (8:00 AM version)                                Sept 28, 2015

Name: Scott Key                                Section: ∞

Closed book and closed notes.                              Answers must include supporting work.
Calculators and cell phones out of sight.

1. (10 pts) Solve the following inequalities:
   (a) \(|2 + 3x| ≤ 3\)
   Method 1
   \[-\frac{2}{3} ≤ x ≤ \frac{1}{3}\]
   Method 2 (two cases)
   \[2 + 3x ≥ 0 \text{ and } 2 + 3x ≤ 3\]
   \[2 + 3x ≥ 0 \text{ and } -(2 + 3x) ≤ 3\]
   \[
   \begin{align*}
   &3x ≥ -2 \quad \text{and} \quad x ≤ \frac{1}{3} \\
   &x ≥ -\frac{2}{3} \quad \text{and} \quad x ≤ \frac{1}{3}
   \end{align*}
   \]
   Solution:
   \[
   \left[-\frac{2}{3} , \frac{1}{3}\right]
   \]
   \[
   \left[-\frac{2}{3} , \frac{1}{3}\right]
   \]
   \[
   \left[-\frac{2}{3} , \frac{1}{3}\right] U \left[-\frac{2}{3} , \frac{1}{3}\right]
   \]
   \[
   \left[-\frac{2}{3} , \frac{1}{3}\right]
   \]
   (b) \(|x - 1| > 3\)
   \[
   \begin{align*}
   &\text{Either} \\
   &x - 1 > 0 \text{ and } x - 1 > 3 \\
   &x - 1 < 0 \text{ and } -(x - 1) > 3
   \end{align*}
   \]
   \[\begin{align*}
   &0 \quad \text{and} \quad x > 4
   \end{align*}\]
   \[\begin{align*}
   &0 \quad \text{and} \quad x < -2
   \end{align*}\]
   Solution:
   \[(-\infty, -2) U (4, \infty)\]

2. (15 pts)
   (a) Determine \(f\) and \(g\) such that \(h(x) = f(g(x))\) for \(h(x) = \sqrt{4x - x^2}\).
   (b) Find the domain of \(h(x)\).

   (a) \[f(x) = \sqrt{x}\]
   \[g(x) = 4x - x^2\]
   \[h(x) = f(g(x)) = \sqrt{4x - x^2}\]

   (b) \[\text{Domain (h) = \{x | 4x - x^2 ≥ 0\} = [0, 4]}\]

   \[4x - x^2 = x(4 - x)\]
   Case 1: \(x ≥ 0 \text{ and } 4 - x ≥ 0\)
   \[x ≥ 0 \text{ and } 4 ≥ x\]
   \[[0, 4]\]
   Case 2: \(x < 0 \text{ and } 4 - x < 0\)
   \[4 < x\]
   No real \(x\) satisfies both \(x < 0\) and \(4 < x\).
3. (10 pts) Given a rectangle, its length is three times its width. Express its area $A$ as a function of its perimeter $P$.

Let $l$ be the length and $w$ the width of the rectangle. $A$ its area and $P$ its perimeter.

\[
\begin{align*}
\text{Area } A &= l \cdot w \\
\text{Length } l &= 3w \\
\text{Perimeter } P &= 2l + 2w = 2(3w) + 2w \\
&= 6w + 2w \\
&= 8w \\
\text{Width } w &= \frac{P}{8} \\
A &= 3w^2 \\
&= 3\left(\frac{P}{8}\right)^2 \\
&= \frac{3P^2}{64} \\
A(P) &= \frac{3P^2}{64}.
\end{align*}
\]

4. (10 pts) The equation $2x^2 - 4x + 2y^2 + 8y + 1 = 0$ describes a circle. Determine the circles center and radius by completing the square.

**Method 1**

\[
\begin{align*}
2x^2 - 4x + 2y^2 + 8y + 1 &= 0 \\
2(x^2 - 2x) + 2(y^2 + 4y) + 1 &= 0 \\
2\left(x^2 - 2x + \frac{1}{2} - \frac{1}{2}\right) + 2\left(y^2 + 4y + 4 - 4\right) + 1 &= 0 \\
2(x^2 - 2x + 1 - 1) + 2(y^2 + 4y + 4 - 4) + 1 &= 0 \\
2\left[(x-1)^2 - 1\right] + 2\left[(y+2)^2 - 4\right] + 1 &= 0 \\
2(x-1)^2 - 2 + 2(y+2)^2 - 8 + 1 &= 0 \\
2(x-1)^2 + 2(y+2)^2 - 9 &= 0 \\
2(x-1)^2 + 2(y+2)^2 &= 9 \\
(x-1)^2 + (y+2)^2 &= \frac{9}{2} = \left(\frac{3}{\sqrt{2}}\right)^2
\end{align*}
\]

**Center:** $(+1, -2)$

**Radius:** $\sqrt{\frac{9}{2}} = \frac{3}{\sqrt{2}}$

**Method 2**

\[
\begin{align*}
2x^2 - 4x + 2y^2 + 8y &= -1 \\
x^2 - 2x + y^2 + 4y &= -\frac{1}{2} \\
x^2 - 2x + \frac{1}{2} + y^2 + 4y + 4 &= -\frac{1}{2} + \frac{1}{2} + 4 \\
(x-1)^2 + (y+2)^2 &= \frac{9}{2} \\
(x-1)^2 + (y+2)^2 &= \left(\frac{3}{\sqrt{2}}\right)^2
\end{align*}
\]
5. (15 pts) Sketch the following graphs:

(a) \( y = |x - 3| + 2 = f(x) \)

(b) \( y = \sqrt{-x} = g(x) \)

(c) \( y = x^2 - 2x + 1 = h(x) \)

\[ f(x) \]

\[ g(x) \]

\[ h(x) \]

6. (10 pts)
(a) Find a polynomial of degree 3 with zeros -3, 0, and 3.
(b) Is \( x - 1 \) a factor of \( P(x) = x^7 + 4x^6 - 2x^5 + x^4 - x^2 + 2x - 5 \)? Explain your answer. [Hint: use the Factor Theorem]

(a) Since -3, 0, and 3 are roots, \((x+3), (x-0), \text{ and } (x-3)\) must be factors. Take:
\[
P(x) = (x+3) \cdot x \cdot (x-3), \quad \text{Full credit}
\]
\[
= x(x+3)(x-3)
\]
\[
= x(x^2-9)
\]
\[
= x^3-9x
\]

(b) Observe:
\[
P(1) = 1^7 + 4 \cdot 1^6 - 2 \cdot 1^5 + 1^4 - 1^2 + 2 \cdot 1 - 5
\]
\[
= 1 + 4 - 2 + 1 - 1 + 2 - 5
\]
\[
= 0
\]
The Factor Theorem tells us that \(x-1\) is a factor of \(P(x)\). Yes.
7. (10 pts) Find the quotient and the remainder using long division.

\[
\begin{array}{c}
\underline{3x^4 - 5x^3 - 20x - 5} \\
\underline{x^2 + x + 3}
\end{array}
\]
\[
\begin{array}{c}
\underline{3x^2 - 8x - 1} \\
3x^4 + 3x^3 + 9x^2
\end{array}
\]
\[
\begin{array}{c}
-8x^3 - 9x^2 - 20x - 5 \\
-8x^3 - 8x^2 - 24x
\end{array}
\]
\[
\begin{array}{c}
-x^2 + 4x - 5 \\
-x^2 - x - 3
\end{array}
\]
\[
5x - 2
\]
\[\text{Quotient: } 3x^2 - 8x - 1 \]
\[\text{Remainder: } 5x - 2\]

8. (10 pts) Write the following polynomial in factored form.

\[P(x) = x^3 + 3x^2 - x - 3\]

Possible integer roots: +1, -1, +3, -3.

\[P(1) = 1 + 3 - 1 - 3 = 0, \text{ so } (x-1) \text{ is a factor.}\]

\[P(-1) = -1 + 3 + 1 - 3 = 0, \text{ so } (x+1) \text{ is a factor.}\]

\[P(-3) = (-3)^3 + 3(-3)^2 - (-3) - 3\]

\[= -27 + 27 + 3 - 3 = 0, \text{ so } (x+3) \text{ is a factor.}\]

\[P(x) = (x-1)(x+1)(x+3)\]

Thus, there were other ways to solve this problem.
9. (10 pts) Sketch the region bounded by the parabola $x = y^2$ and the line $y = x - 2$
and label their points of intersection.

\[ x = (x-2)^2 \]
\[ = x^2 - 4x + 4 \]
\[ 0 = x^2 - 5x + 4 \]
\[ = (x-4)(x-1) \]
\[ x = 4 \quad \text{or} \quad x = 1 \]
\[ y = x - 2 \quad y = 1 - 2 \]
\[ = 2 \quad = -1 \]

\((4,2)\) \quad \((1,-1)\) \hspace{1cm} \text{Intersection points, i.e., points that lie on both equations' graphs.}
Math 223  
Midterm (11:20 AM version)  
Sept 28, 2015

Name: Frank Key

Closed book and closed notes.  
Calculators and cell phones out of sight.

Answers must include supporting work.

1. (10 pts) Solve the following inequalities:
(a) \(|1 - 2x| \leq 3\)
(b) \(|x - 2| > 1\)

\[\begin{align*}
\text{Method 1} & \quad \text{Method 2} \\
\text{Case 1:} & \quad -2x \geq 1 \text{ and } -2x \leq 2 \\
& \quad \pm \quad -x \geq -\frac{1}{2} \quad \pm \quad x \leq 1 \\
& \quad \text{or} \quad [1, \frac{3}{2}] \\
\text{Case 2:} & \quad 1 - 2x < 0 \text{ and } -1 - 2x \leq 3 \\
& \quad 1 - 2x < -1 \quad \pm \quad -2x \leq 2 \\
& \quad x \leq \frac{1}{2} \quad \pm \quad x \leq 1 \\
& \quad \left(\frac{1}{2}, 2\right] \\
\text{Solution:} & \quad [-1, \frac{1}{2}] \cup \left(\frac{1}{2}, 2\right] \\
\text{Case 1:} & \quad x - 2 \geq 0 \text{ and } x - 2 > 1 \\
& \quad x \geq 2 \text{ and } x > 3 \\
& \quad \text{or} \quad [3, \infty) \\
\text{Case 2:} & \quad x - 2 < 0 \text{ and } -(x - 2) > 1 \\
& \quad x < 2 \text{ and } x - 2 < -1 \\
& \quad \text{or} \quad (-\infty, 1) \\
\text{Solution:} & \quad (-\infty, 1) \cup (3, \infty).
\end{align*}\]

2. (15 pts)
(a) Determine \(f\) and \(g\) such that \(h(x) = f(g(x))\) for \(h(x) = \sqrt{2x - x^2}\).
(b) Find the domain of \(h(x)\).

\[\begin{align*}
\text{(a)} & \quad g(x) = 2x - x^2 \\
& \quad f(x) = \sqrt{x} \\
& \quad h(x) = f(g(x)) = \sqrt{2x - x^2} \\
\text{(b)} & \quad \text{dom}(h) = \{x \mid 2x - x^2 \geq 0\} \\
& \quad 2x - x^2 \geq 0 \\
& \quad x(2 - x) \geq 0 \\
& \quad \text{Case 1:} \quad x \geq 0 \text{ and } 2 - x \geq 0 \quad \text{Case 2:} \quad x < 0 \text{ and } 2 - x < 0 \\
& \quad x \geq 0 \quad \text{or} \quad 2 \geq x \\
& \quad 0 \leq x \leq 2 \\
& \quad [0, 2]
\end{align*}\]
3. (10 pts) A rectangle has perimeter 12 m. Express the area $A$ of the rectangle as a function of the length, $L$, of one of its sides.

\[
\begin{align*}
L & \quad w \\
L \quad W & = 2L + 2W = 12, \\
A & = L \cdot W.
\end{align*}
\]

Let $L$ be the length of one side, $W$ the length of the other, $A$ the area and $P$ the perimeter.

\[
\begin{align*}
A(L) & = L(6-L) \\
2W & = 12 - 2L \\
W & = \frac{1}{2}(12-2L) \\
& = 6-L
\end{align*}
\]

$\Leftarrow$ Expanding not required and often discouraged.

4. (10 pts) The equation $x^2 + y^2 - 4x + 10y + 13 = 0$ describes a circle. Determine the circles center and radius by completing the square.

**Method 1**

\[
\begin{align*}
x^2 - 4x + y^2 + 10y & = -13 \\
x^2 - 4x + 4 + y^2 + 10y + 25 & = -13 + 4 + 25 \\
(x-2)^2 + (y+5)^2 & = 16 = 4^2
\end{align*}
\]

Center: $(2, -5)$
Radius: $4$

**Method 2**

\[
\begin{align*}
x^2 + y^2 - 4x + 10y + 13 & = 0 \\
x^2 - 4x + 4 + y^2 + 10y + 25 - 25 + 13 & = 0 \\
(x-2)^2 + (y+5)^2 - 4 - 25 + 13 & = 0 \\
(x-2)^2 + (y+5)^2 & = 16
\end{align*}
\]

Center: $(2, -5)$
Radius: $4$
5. (15 pts) Sketch the following functions:

(a) \( f(x) = \begin{cases} x + 3 & \text{if } x < 0 \\ 1 - x & \text{if } x \geq 0 \end{cases} \)

(b) \( y = -\sqrt{x} = g(x) \)

(c) \( y = x^2 - 4x + 4 = h(x) \)

6. (10 pts)

(a) Find a polynomial of degree 3 with zeros -2, 0, and 2.

(b) Is \( x - 1 \) a factor of \( P(x) = x^7 + 4x^6 - 2x^5 + x^4 - x^2 + 2x - 5 \)? Explain your answer. [Hint: use the Factor Theorem]

By the Factor Theorem, \( (x - (-2)), (x - 0), \) and \( (x - 2) \) must be factors.

\[ P(x) = (x + 2)(x - 2) \cdot (x - 0)(x - 2) \]

This clearly has degree 3.

\[ = (x + 2) \cdot x \cdot (x - 2) \]

\[ = x^3 - 4x \]

\[ = x^3 - 4x \]

\[ \{ \text{Not needed; usually more complicated.} \} \]

(b) Since \( P(1) = 1^7 + 4 \cdot 1^6 - 2 \cdot 1^5 + 1^4 - 1^2 + 2 \cdot 1 - 5 \)

\[ = 1 + 4 - 2 + 1 - 1 + 2 - 5 \]

\[ = 0 \]

by the Factor Theorem \( (x - 1) \) is a factor of \( p(x) \). Yes.
7. (10 pts) Find the quotient and the remainder using long division.

\[
\begin{array}{c|c}
3x^4 - 5x^3 - 20x - 5 \\
x^2 + x + 3 \\
\hline \\
3x^2 - 8x - 1 & \text{Quotient: } 3x^2 - 8x - 1 \\
3x^4 + 3x^3 + 9x^2 \\
\hline \\
-8x^3 - 9x^2 - 20x - 5 \\
-8x^3 - 8x^2 - 14x \\
\hline \\
-x^2 + 14x - 5 \\
-x^2 - x - 3 \\
\hline \\
5x - 2 & \text{Remainder: } 5x - 2
\end{array}
\]

8. (10 pts) Factor completely the following polynomial.

\[P(x) = x^3 + 3x^2 - x - 3\]

Possible integer roots: \(1, -1, 3, -3\).

Observe: 
\[P(1) = 1^3 + 3 \cdot 1^2 - 1 - 3 = 1 + 3 - 1 - 3 = 0 \quad \text{so } (x-1) \text{ is a factor.}\]
\[P(-1) = (-1)^3 + 3 \cdot (-1)^2 - (-1) - 3 = (-1) + 3 + 1 - 3 = 0 \quad \text{so}\]
\[P(-3) = (-3)^3 + 3 \cdot (-3)^2 - (-3) - 3 = -27 + 27 + 3 - 3 = 0 \quad \text{so } (x-(-3)) = (x+3) \text{ is a factor.}\]

\[P(x) = (x-1)(x+1)(x+3).\]

There were other ways to solve this problem.
9. (10 pts) Sketch the region bounded by the parabola \( x = y^2 \) and the line \( y = 2 - x \) and label their points of intersection.

\[ \begin{align*}
    y &= 2 - x \\
    x &= y^2 = (2-x)^2 \\
    &= 4 - 4x + x^2 \\
    x^2 - 4x + 4 &= x \\
    x^2 - 5x + 4 &= 0 \\
    (x-4)(x-1) &= 0 \\
    x &= 4 \text{ or } x = 1
\end{align*} \]

Curves meet when \( x = 4 \) and when \( x = 1 \):

\[ \begin{align*}
    y &= 2 - 4 \\
    &= -2 \\
    y &= 2 - 1 \\
    &= 1
\end{align*} \]

$$(4, -2) \quad (1, 1)$$

Intersection points - points that lie on the graphs of both equations.