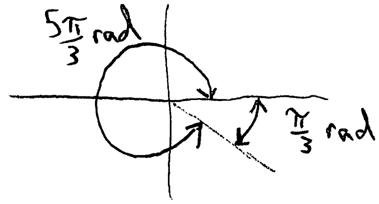
Math 223 Final Exam Oct 23, 2015

Solution set

(1.) $\theta = 300° = 300° \left(\dfrac{\pi \text{ radians}}{180°}\right) = \dfrac{5\pi}{3}$ radians.



$$\sin\left(\frac{5\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right)$$
$$= -\frac{\sqrt{3}}{2}.$$

$$\cos\left(\frac{5\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}.$$

$$\tan\left(\frac{5\pi}{3}\right) = \frac{\sin\left(\frac{5\pi}{3}\right)}{\cos\left(\frac{5\pi}{3}\right)} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\frac{\sqrt{3}}{2}\cdot\frac{2}{1} = -\sqrt{3}.$$

(2.) $$2\ln(x) + 3\ln(y) - \ln(z) = \ln(x^2) + \ln(y^3) - \ln(z)$$
$$= \ln\left(\frac{x^2 y^3}{z}\right).$$

(3.) a.) $x = \log_8 2$ means $8^x = 2$, so $x = \frac{1}{3}$.

b.) $$\log_5 \frac{1}{125} = \log_5(125^{-1})$$
$$= -\log_5(125)$$
$$= -\log_5(5^3)$$
$$= -3\log_5(5)$$
$$= -3\cdot 1$$
$$= 3.$$

(4) $f(x) = \ln(6 - x - x^2)$. This function is defined whenever $6 - x - x^2 > 0$ because the log function is defined only for positive input. So the domain of $f(x)$ is the set of x values satisfying $6 - x - x^2 > 0$.

Solving the inequality, we have:

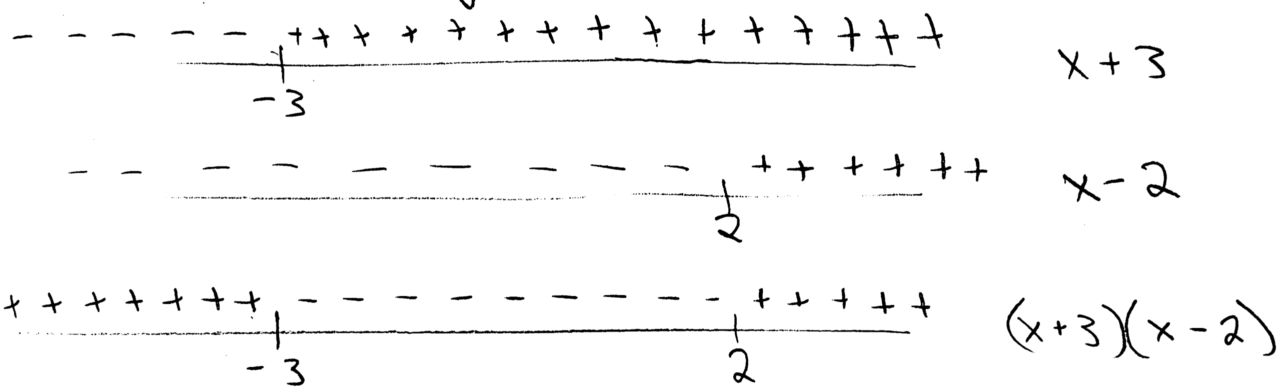
$$6 - x - x^2 > 0$$

$$\Rightarrow \quad -6 + x + x^2 < 0 \qquad \text{(multiplying by } -1\text{)}$$

$$\Rightarrow \quad x^2 + x - 6 < 0$$

$$\Rightarrow \quad (x+3)(x-2) < 0$$

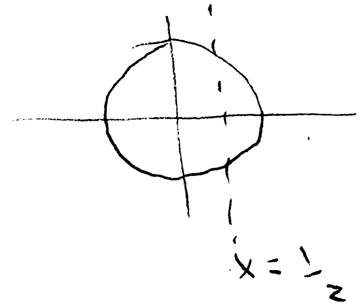
This product is negative when $x+3$ and $x-2$ have different signs.



So the domain of $f(x)$ is the open interval $(-3, 2)$.

5. (a.) $2\cos x - 1 = 0$

$\Rightarrow \quad \cos x = \frac{1}{2}$.



So x is an angle whose cosine is $\frac{1}{2}$. In quadran 1, we find $x = \frac{\pi}{3}$ radians. In quadrant 4, we find $x = \frac{5\pi}{3}$ radians. So, $x = \frac{\pi}{3}$ or $x = \frac{5\pi}{3}$ radians.

(b.) $e^{7-4x} = 6$ (take ln of both sides)

$\Rightarrow \quad 7 - 4x = \ln 6$

$\Rightarrow \quad -4x = \ln 6 - 7$

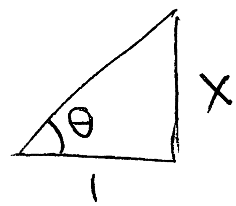
$\Rightarrow \quad x = \dfrac{\ln 6 - 7}{-4} = \dfrac{7 - \ln 6}{4}$.

(c.) $e^{e^x} = 10$ (take ln of both sides)

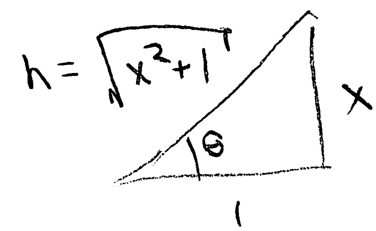
$\Rightarrow \quad e^x = \ln 10$ (do it again)

$\Rightarrow \quad x = \ln(\ln 10)$.

(6.) $\tan^{-1}(x)$ is an angle. Let's call the angle θ. So $\theta = \tan^{-1}(x)$. In other words, θ is an angle whose tangent is x, or equivalently, $\frac{x}{1}$. We can fill-in a triangle with this info:



and we can use the Pythagorean Theorem to get the hypotenuse:



$$h^2 = x^2 + 1^2$$

$$\Rightarrow h = \pm\sqrt{x^2+1}$$ and

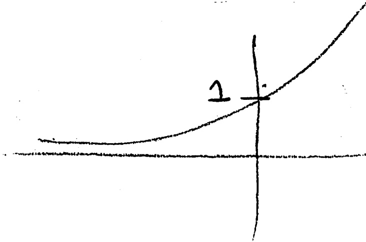
we use only positive root since h is a distance

From the triangle, we now just write down

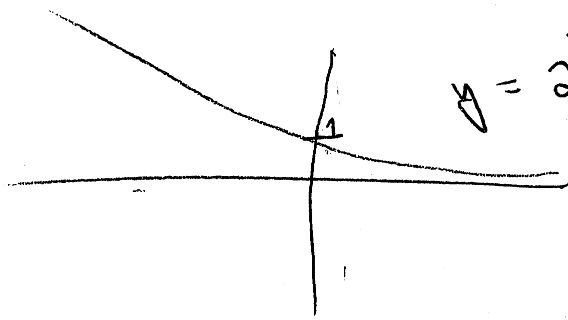
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{x}{\sqrt{x^2+1}}.$$

So $$\sin(\tan^{-1} x) = \sin\theta = \frac{x}{\sqrt{x^2+1}}.$$

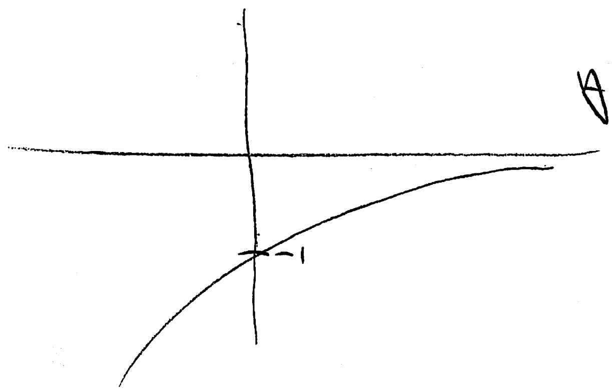
7. (a.)



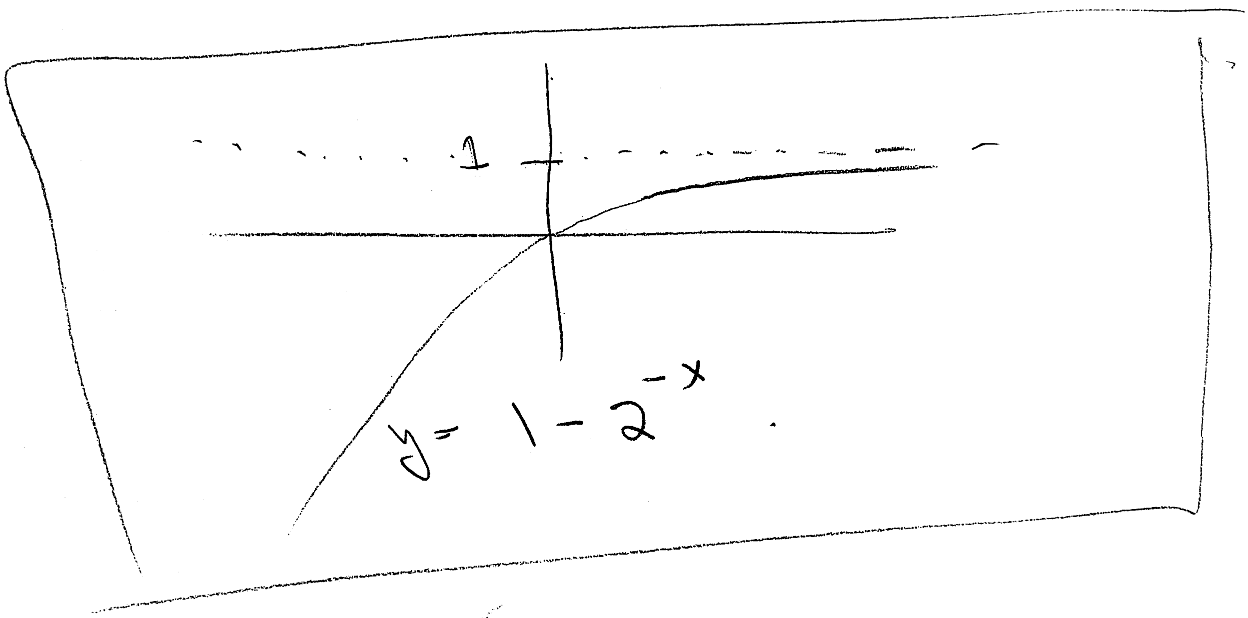
$y = 2^x$



$y = 2^{-x}$

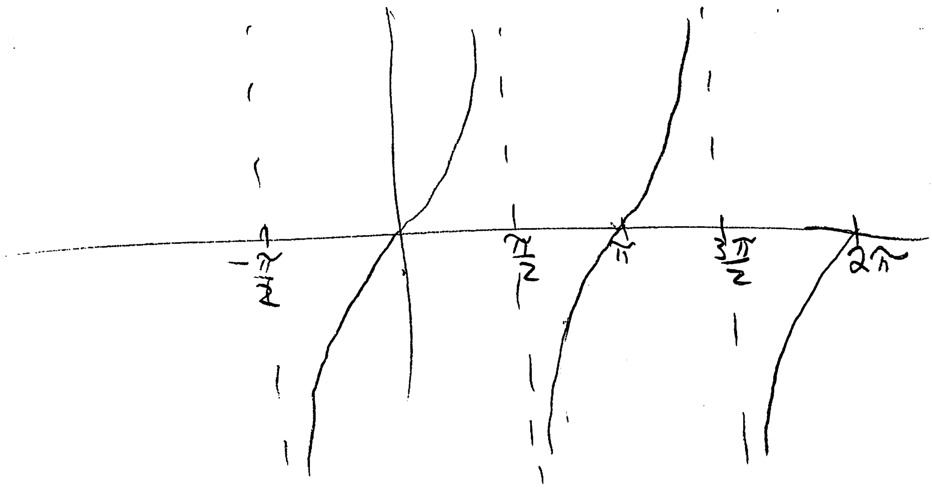


$y = -2^{-x}$

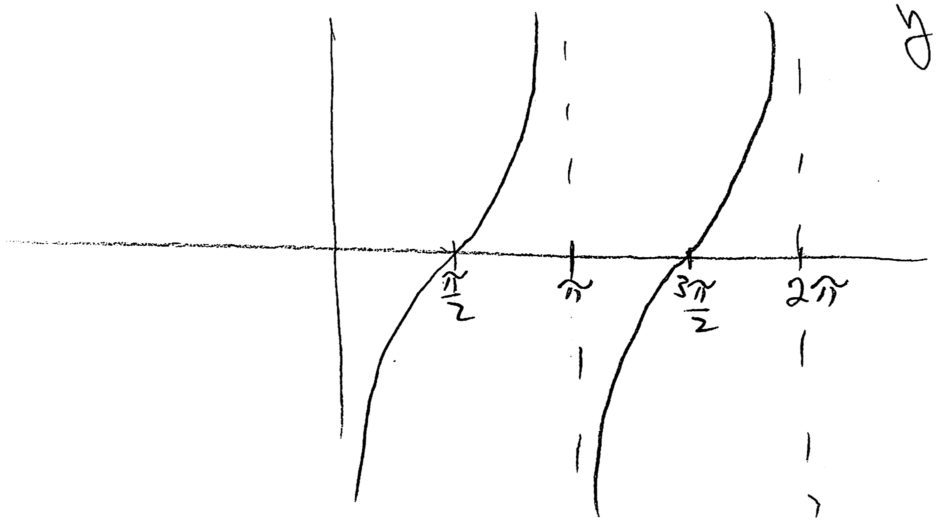


$y = 1 - 2^{-x}$

⑦ (b)



$y = \tan x$



$y = \tan\left(x - \frac{\pi}{2}\right)$

(~~shifts~~ shift right by $\frac{\pi}{2}$ units)

8. All three pieces are polynomials, which are continuous everywhere, so discontinuities can only happen at $x=0$ and $x=1$. Let's check each location.

$\underline{x=0}$: $\lim_{x\to 0^-} f(x) = 0+2 = 2$

$\lim_{x\to 0^+} f(x) = 2(0)^2 = 0$

$\underline{x=1}$: $\lim_{x\to 1^-} f(x) = 2(1)^2 = 2$

$\lim_{x\to 1^+} f(x) = 3-1 = 2.$

The two-sided limit of $f(x)$ does not exist at $x=0$ because $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$. So $f(x)$ is not continuous at $x=0$.

The two-sided limit of $f(x)$ exists at $x=1$ because $\lim_{x\to 1^-} f(x) = 2 = \lim_{x\to 1^+} f(x)$. And the value of $\lim_{x\to 1} f(x)$ is equal to $f(1)$. So $f(x)$ is continuous at $x=1$.

Hence, $f(x)$ is discontinuous only at $x=0$.

9 (a.) $\lim_{x \to 3} \frac{x^2-9}{x^2-x-6} = \lim_{x \to 3} \frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}(x+2)}$

$= \lim_{x \to 3} \frac{x+3}{x+2} = \frac{6}{5}$.

(b.) $\lim_{x \to 4^-} \frac{x^2-16}{|x-4|} = \lim_{x \to 4^-} \frac{(x+4)(x-4)}{|x-4|}$.

Since x approaches 4 from the left, $x-4$ is negative, $|x-4|$ is positive, and $x+4$ is positive. So, $\frac{(x+4)(x-4)}{|x-4|}$ is negative.

Because $x-4$ is negative, we know that $|x-4| = -(x-4)$. This gives us:

$$\lim_{x \to 4^-} \left[\frac{(x+4)\cancel{(x-4)}}{-\cancel{(x-4)}}\right] = \lim_{x \to 4^-} [-(x+4)]$$

$$= -8.$$

9. (c.) $\lim_{h\to 0} \dfrac{\sqrt{9+h}-3}{h}$

$= \lim_{h\to 0} \dfrac{\sqrt{9+h}-3}{h}\left(\dfrac{\sqrt{9+h}+3}{\sqrt{9+h}+3}\right)$

$= \lim_{h\to 0} \dfrac{(\cancel{9}+h) - \cancel{3^2}}{h(\sqrt{9+h}+3)}$ ← (difference of two squares)

$= \lim_{h\to 0} \dfrac{\cancel{h}}{\cancel{h}(\sqrt{9+h}+3)}$

$= \lim_{h\to 0} \dfrac{1}{\sqrt{9+h}+3}$

$= \dfrac{1}{\sqrt{9+0}+3} = \dfrac{1}{3+3} = \dfrac{1}{6}.$