Math 223: Final Exam (Version 1)
10/18/2021

Name: Solutions

Section: ______

INSTRUCTIONS:

There are a total of 100 points. You must turn off and put away all electronic devices and you may not look at the textbook or any form or written notes. All answers must include supporting work. Please begin the exam only when your instructor tells you.

Problem:	1	2	3	4	5	6	7	8	9	Total
Points:	10	12	10	8	15	15	10	10	10	100
Your Score:										

1. (10 pts) Find $f^{-1}(x)$, given that $f(x) = 8 + \sqrt{4+3x}$ is one-to-one.

$$y = 8 + \sqrt{4+3x}$$

$$y - 8 = \sqrt{4+3x}$$

$$(y-8)^2 = 4+3x$$

$$x = \frac{(y-8)^2 - 4}{3}$$

$$\boxed{f^{-1}(x) = \frac{1}{3}(x-8)^2 - \frac{4}{3}}$$

* Two ways to do part (a). *

2. (12 pts) Find all x-values that satisfy the following equations.

a) $5^{2+3x} = 2$

$$\ln(5^{2+3x}) = \ln 2 \quad \text{"or"} \quad \log_5(5^{2+3x}) = \log_5(2)$$

$$(2+3x)\ln 5 = \ln 2 \qquad 2+3x = \log_5(2)$$

$$2+3x = \frac{\ln 2}{\ln 5} \qquad 3x = \log_5(2) - 2$$

$$\boxed{x = \frac{\ln 2}{3\ln 5} - \frac{2}{3}} \quad \text{"or"} \quad \boxed{x = \frac{1}{3}\log_5(2) - \frac{2}{3}}$$

b) $\ln(5x+7) = 4$

$$5x + 7 = e^4$$

$$5x = e^4 - 7$$

$$\boxed{x = \frac{e^4 - 7}{5}}$$

c) $\log_2(x) + \log_2(x+2) = 3$

$$\log_2[x(x+2)] = 3$$

$$x(x+2) = 2^3$$

$$x^2 + 2x = 8$$

$$x^2 + 2x - 8 = 0$$

$$(x+4)(x-2) = 0 \Rightarrow \boxed{x = -4, 2}$$

3. (10 pts) Express the following quantity as a single logarithm.

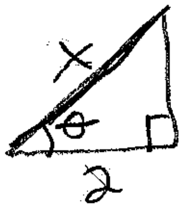
$$\frac{1}{4}\ln(x^8) + 2\ln(x+2) - 3\ln(x+6) - \frac{1}{2}\ln(4)$$

$$\ln(x^2) + \ln[(x+2)^2] - \ln[(x+6)^3] - \ln(2)$$

$$\ln\left(\frac{x^2(x+2)^2}{(x+6)^3(2)}\right)$$

4. (8 pts) Simplify the expression $\tan\left(\cos^{-1}\left(\frac{2}{x}\right)\right)$.

$$\text{Let } \theta = \cos^{-1}\left(\tfrac{2}{x}\right) \Leftrightarrow \frac{2}{x} = \cos\theta$$



$u = \sqrt{x^2-4}$

$$2^2 + u^2 = x^2$$

$$u = \sqrt{x^2-4}$$

$$\tan\left(\cos^{-1}\left(\tfrac{2}{x}\right)\right) = \tan\theta = \boxed{\frac{\sqrt{x^2-4}}{2}}$$

5. (15 pts) Calculate the following limits. If the limit does not exist, explain why. You may not use L'Hopital's rule.

a) $\lim_{x \to 5^-} \frac{x-5}{x^2-10x+25}$ $= \lim_{x\to 5^-} \frac{x-5}{(x-5)(x-5)}$

$= \lim_{x\to 5^-} \frac{1}{x-5}$

$= \boxed{-\infty}$ DNE

b) $\lim_{x \to 3^-} \frac{x^2-9}{|x-3|}$

$|x-3| = \begin{cases} x-3, & x-3 \geq 0 \\ -(x-3), & x-3 < 0 \end{cases}$

$= \begin{cases} x-3, & x \geq 3 \\ -(x-3), & x < 3 \end{cases}$

$= \lim_{x\to 3^-} \frac{(x-3)(x+3)}{-(x-3)}$

$= \lim_{x\to 3^-} -(x+3)$

$= \boxed{-6}$

c) $\lim_{x \to -2} \frac{\sqrt{x^2+5}-3}{x+2} \cdot \frac{\sqrt{x^2+5}+3}{\sqrt{x^2+5}+3}$

$= \lim_{x\to -2} \frac{x^2+5-9}{(x+2)(\sqrt{x^2+5}+3)} = \lim_{x\to -2} \frac{x^2-4}{(x+2)(\sqrt{x^2+5}+3)}$

$= \lim_{x\to -2} \frac{(x-2)(x+2)}{(x+2)(\sqrt{x^2+5}+3)} = \lim_{x\to -2} \frac{x-2}{\sqrt{x^2+5}+3}$

$= \frac{-4}{6} = \boxed{\frac{-2}{3}}$

6. (15 pts)

a) Fill in the blank to complete the definition of a **continuous function.**

A function f is continuous at a number a if $\lim_{x \to a} f(x) = \underline{f(a)}$.

b) Find all x-values at which the following function is **discontinuous.** You must justify your answers fully using the definition of continuity to receive full credit.

$$f(x) = \begin{cases} 1/x & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 2 - x & \text{if } 1 < x \le 4 \\ \sqrt{x} - 3 & \text{if } x > 4 \end{cases}$$

$f(1) = 3$

$f(4) = -2$

$\lim_{x \to 1^-} f(x) = 1$

$\lim_{x \to 1^+} f(x) = 1$

$\lim_{x \to 4^-} f(x) = -2$

$\lim_{x \to 4^+} f(x) = -1$

Discont. @ $x = 1$ due to $\lim_{x \to 1} f(x) = 1 \neq f(1) = 3$

Discont. @ $x = 4$ due to $\lim_{x \to 4^-} f(x) \neq \lim_{x \to 4^+} f(x)$

c) Now consider the function $g(x) = \begin{cases} \sqrt{cx} & \text{if } 0 \le x \le 3 \\ x^2 + 1 & \text{if } x > 3 \end{cases}$

For what value of the constant c is the function g continuous at $x = 3$?

$\lim_{x \to 3^-} g(x) = \lim_{x \to 3^+} g(x) = g(3)$

$\sqrt{3c} = 10$

$3c = 100$

$\boxed{c = \frac{100}{3}}$

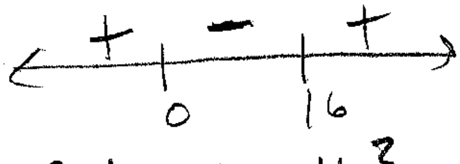
7. (10 pts) Find the domain of the following function:

$$f(x) = \ln(16x - x^2) + e^x$$

$16x - x^2 > 0$

$x^2 - 16x < 0$

$x(x-16) < 0$



$D = \{x \mid 0 < x < 16\}$

$= \boxed{(0,16)}$

8. (10 pts) Find the exact value of each expression:

a) $\log_4(1600) - 2\log_4(10)$ $= \log_4\left(\frac{1600}{100}\right) = \log_4(16) = \boxed{2}$

b) $\sin^{-1}\left(\frac{1}{2}\right) = \theta \Leftrightarrow \sin\theta = \frac{1}{2}$

$\boxed{\theta = \frac{\pi}{6}}$

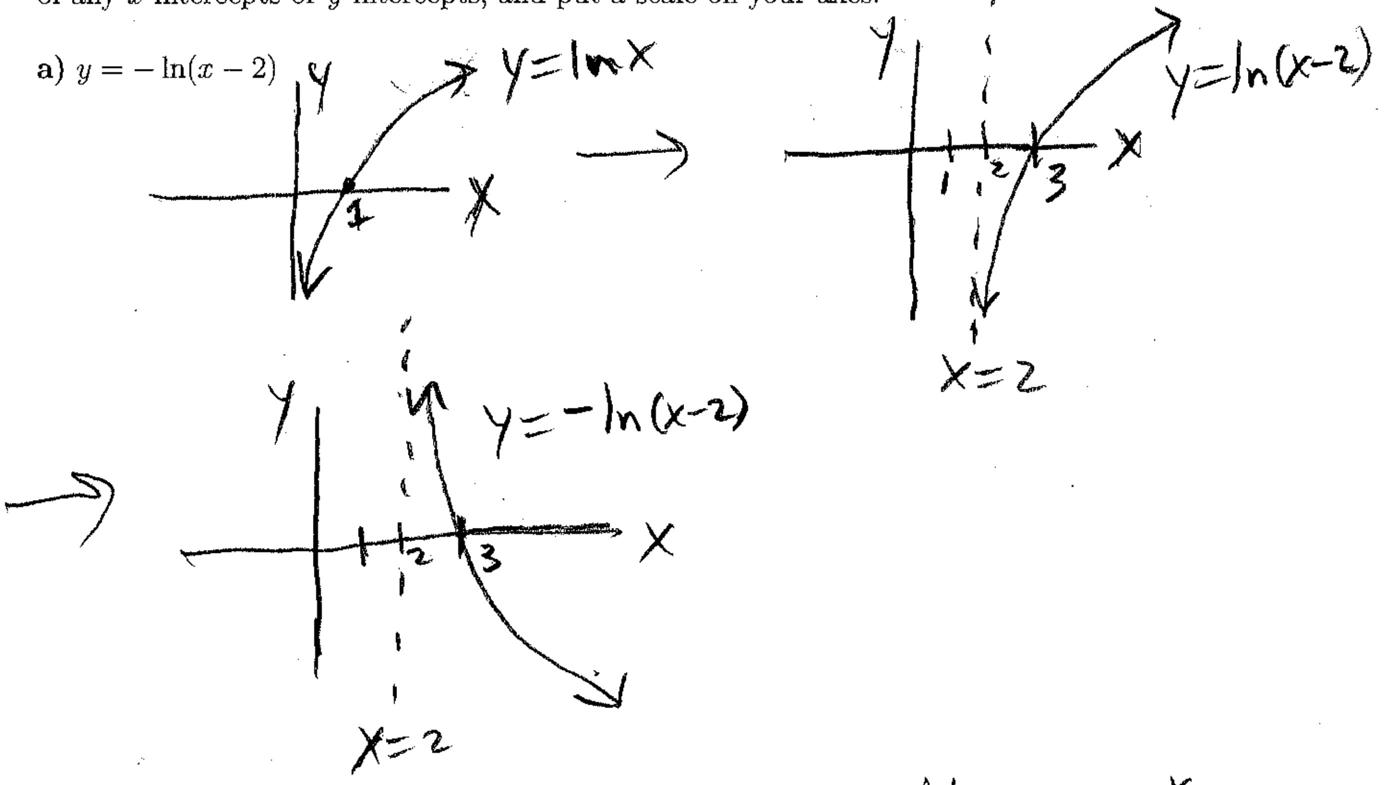
c) $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \theta \Leftrightarrow \cos\theta = \frac{-1}{\sqrt{2}} \Rightarrow \theta = \boxed{\frac{3\pi}{4}}$

d) $\arcsin\left(\sin\left(\frac{5\pi}{4}\right)\right)$ $= \arcsin\left(-\frac{1}{\sqrt{2}}\right) = \boxed{\frac{-\pi}{4}}$

e) $4^{\log_4(31)}$ $= \boxed{31}$

9. (10 pts) Sketch the graphs of the following functions. For full credit, label the coordinates of any x-intercepts or y-intercepts, and put a scale on your axes.

a) $y = -\ln(x-2)$



b) $y = -2^{-x} + 2$